An Angle-Free Lorentz Embedding for Directional Separation

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Abstract

This note records a compact angle-free representation of directional separation in a Lorentzian embedding space. Instead of treating angular coordinates as primitive, each direction is represented by a unit vector. Relative direction is then carried by an algebraic dot product, and separation is carried by a Lorentzian inner-product invariant. The same algebra also gives a clean reading of coordinate time: time is the zeroth Lorentz coordinate in a chosen frame, while invariant separation remains an inner-product statement. The final sections extend the same principle from one closed direction to a no-angle torus, then to a hopfion-like linked-fibre support. The construction is coordinate-clean, dimension-independent, and does not require angular parametrisation.

1 Introduction

In many geometric descriptions, a relative direction is written by introducing an angular coordinate. That is often convenient, but it is not necessary. The same information can be represented algebraically by assigning each state a unit direction vector and using the dot product between those unit vectors.

This note gives the minimal construction. The aim is not to introduce a new physical law, but to show that angular variables can be treated as optional coordinates rather than primitive data. The primitive data are an embedded vector, a unit direction, and a Lorentzian inner product, in the standard setting of Minkowski and Lorentzian geometry [1, 2, 3].

2 No-Angle Replacement Principle

The angular term can be removed by replacing the angle difference with an algebraic dot product between unit direction vectors. Instead of writing the directional alignment as

$$\cos(\theta_2 - \theta_1), \tag{1}$$

one may write

$$u_1 \cdot u_2, \tag{2}$$

with

$$u_i \cdot u_i = 1. \tag{3}$$

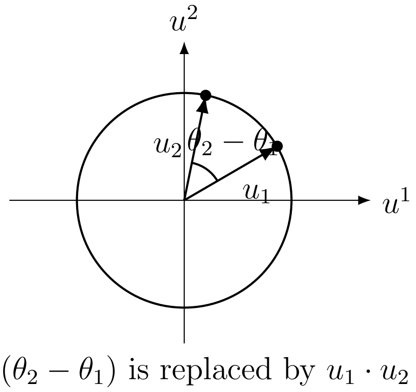


Figure 1: Angle-free directional alignment. The angular coordinate is optional because the relative alignment of two unit directions is already carried by $u_1 \cdot u_2$.

The hyperbolic separation is then carried by the invariant

$$\chi_{12} = C_1 C_2 - S_1 S_2 (u_1 \cdot u_2), \tag{4}$$

where

$$C_i^2 - S_i^2 = 1, \tag{5}$$

and

$$X_i = (C_i, S_i u_i) \tag{6}$$

satisfies

$$\langle X_i, X_i \rangle_\eta = 1. \tag{7}$$

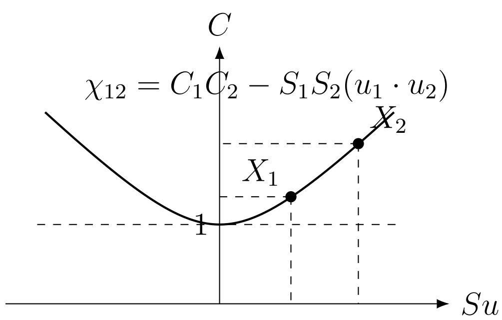


Figure 2: Lorentzian invariant without angular coordinates. The comparison is made by the inner product of embedded states, with direction entering through $u_1 \cdot u_2$.

In higher dimensions, nothing essential changes. Let

$$u_i \in \mathbb{R}^n, \tag{8}$$

with

$$\sum_{a=1}^{n} (u_i^a)^2 = 1. \tag{9}$$

Then

$$\chi_{12} = C_1 C_2 - S_1 S_2 \sum_{a=1}^{n} u_1^a u_2^a. \tag{10}$$

Thus the no-trigonometric form is not a weaker approximation. It is the same geometry expressed through embedded unit directions and Lorentzian inner products rather than angular parametrisation.

When a standard unit-hyperboloid distance convention is introduced, the convention used in hyperbolic geometry gives [4, 5]

$$\cosh(d_{12}) = \chi_{12}. \tag{11}$$

If one wants to avoid inverse functions, then χ_{12} may be used directly as the distance-ordering invariant:

$$d_{12} \sim \chi_{12}. \tag{12}$$

The point is that angular variables are not primitive data:

$$\theta_1, \theta_2 \notin \text{primitive data}. \tag{13}$$

The circular angular expression is only one coordinate representation of the algebraic alignment $u_1 \cdot u_2$.

3 Two-Dimensional Direction Space

The same construction may be written with notation chosen to avoid collision with survival weights, counts, or other model-specific symbols. Let u_i be a unit direction vector in $\mathbb{R}^2$:

$$u_i = (u_i^1, u_i^2), \tag{14}$$

with

$$u_i \cdot u_i = (u_i^1)^2 + (u_i^2)^2 = 1. \tag{15}$$

Let α_i and ρ_i be scalar embedding coordinates satisfying

$$\alpha_i^2 - \rho_i^2 = 1. \tag{16}$$

Define the embedded vector

$$X_i = (\alpha_i, \rho_i u_i^1, \rho_i u_i^2). \tag{17}$$

Use the Lorentzian metric

$$\eta = \mathrm{diag}(1, -1, -1). \tag{18}$$

The Lorentzian norm of X_i is

$$\langle X_i, X_i \rangle_\eta = \alpha_i^2 - \rho_i^2 (u_i \cdot u_i). \tag{19}$$

Using (15) and (16),

$$\langle X_i, X_i \rangle_\eta = 1. \tag{20}$$

For two embedded states X_1 and X_2, define the invariant

$$\chi_{12} = \langle X_1, X_2 \rangle_\eta. \tag{21}$$

Expanding gives

$$\chi_{12} = \alpha_1\alpha_2 - \rho_1\rho_2(u_1 \cdot u_2). \tag{22}$$

Equivalently,

$$\chi_{12} = \alpha_1\alpha_2 - \rho_1\rho_2\left(u_1^1 u_2^1 + u_1^2 u_2^2\right). \tag{23}$$

Thus relative direction enters only through the dot product $u_1 \cdot u_2$. No angular variable is required.

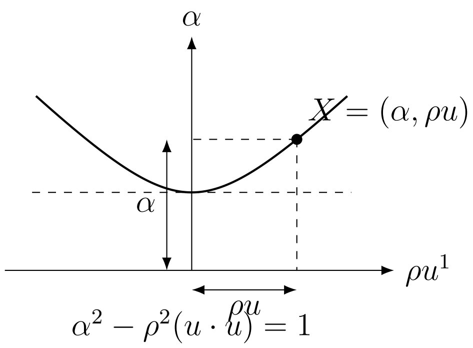


Figure 3: Two-dimensional embedded state. The unit direction u controls orientation, while ρ gives spatial size and α gives the time-like coordinate.

4 Higher-Dimensional Version

The same construction works in n spatial embedding dimensions. Let

$$u_i = (u_i^1, \ldots, u_i^n) \in \mathbb{R}^n, \tag{24}$$

with

$$\sum_{a=1}^{n} (u_i^a)^2 = 1. \tag{25}$$

Let

$$\alpha_i^2 - \rho_i^2 = 1. \tag{26}$$

Define

$$X_i = (\alpha_i, \rho_i u_i^1, \ldots, \rho_i u_i^n) \in \mathbb{R}^{1,n}. \tag{27}$$

With

$$\eta = \mathrm{diag}(1, -1, \ldots, -1), \tag{28}$$

the unit-embedding condition is

$$\langle X_i, X_i \rangle_\eta = \alpha_i^2 - \rho_i^2 \sum_{a=1}^{n} (u_i^a)^2 = 1. \tag{29}$$

The pairwise invariant is

$$\chi_{12} = \langle X_1, X_2 \rangle_\eta = \alpha_1\alpha_2 - \rho_1\rho_2 \sum_{a=1}^{n} u_1^a u_2^a. \tag{30}$$

This invariant orders separations in the embedded Lorentzian geometry. The directional part remains purely algebraic.

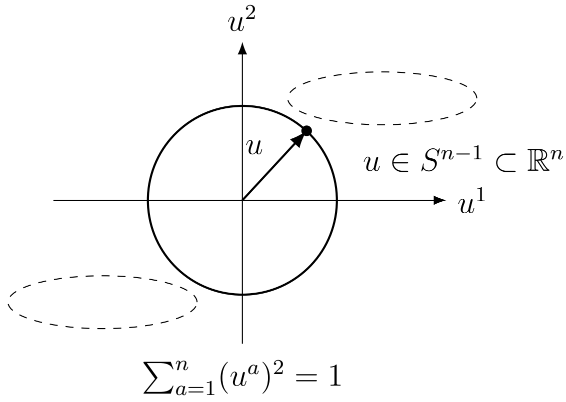


Figure 4: Higher-dimensional direction shell. In any dimension, the direction data live on the unit sphere S^{n-1}, and relative direction is still the dot product $\sum_a u_1^a u_2^a$.

5 Coordinate-Only Form

The construction can also be written without separating radial and direction data. Let

$$X_i = (X_i^0, X_i^1, \dots, X_i^n), \tag{31}$$

with

$$(X_i^0)^2 - \sum_{a=1}^{n}(X_i^a)^2 = 1, \qquad X_i^0 > 0. \tag{32}$$

Then the invariant is simply

$$\chi_{12} = X_1^0 X_2^0 - \sum_{a=1}^{n} X_1^a X_2^a. \tag{33}$$

This coordinate-only version makes clear that no angular coordinates are part of the primitive data. If desired, unit directions can be recovered from the spatial components after a non-zero spatial magnitude has been specified, but the invariant itself does not require that recovery. This is the same discipline used in the associated no-trigonometric recursive phase notes [9, 10].

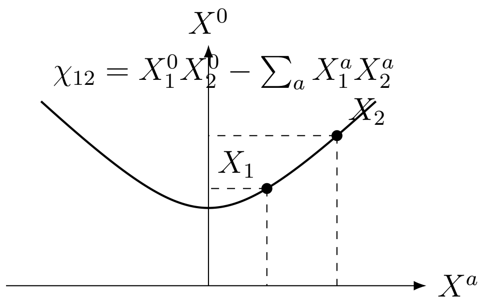


Figure 5: Coordinate-only embedding. The invariant can be computed directly from coordinates without first recovering angles or unit directions.

6 Coordinate Time

The same algebra also explains coordinate time. In the dimensionless embedding above, the zeroth coordinate is

$$X_i^0 = \alpha_i. \tag{34}$$

If a physical time scale τ_0 is chosen, the corresponding physical spacetime vector may be written as

$$Z_i = c\tau_0 X_i = (ct_i, x_i^1, \dots, x_i^n), \tag{35}$$

so that

$$ct_i = c\tau_0 X_i^0, \qquad t_i = \tau_0 X_i^0. \tag{36}$$

Thus coordinate time is not an angular variable. It is the time projection of the Lorentz vector after a frame and a scale have been chosen.

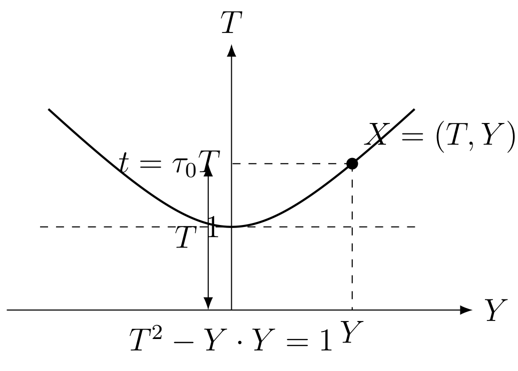


Figure 6: Coordinate time as projection. The time coordinate is read from the zeroth Lorentz coordinate T, then scaled by $t = \tau_0 T$.

In the separated form

$$X_i = (T_i, R_i u_i), \qquad u_i \cdot u_i = 1, \tag{37}$$

the unit hyperboloid condition is

$$T_i^2 - R_i^2 = 1. \tag{38}$$

On the future sheet,

$$T_i = \sqrt{1 + R_i^2}, \qquad t_i = \tau_0 \sqrt{1 + R_i^2}. \tag{39}$$

This is the time analogue of the no-angle construction. The radial spatial size R_i and the direction u_i determine the Lorentz vector algebraically, without introducing rapidity or any angular coordinate as primitive data.

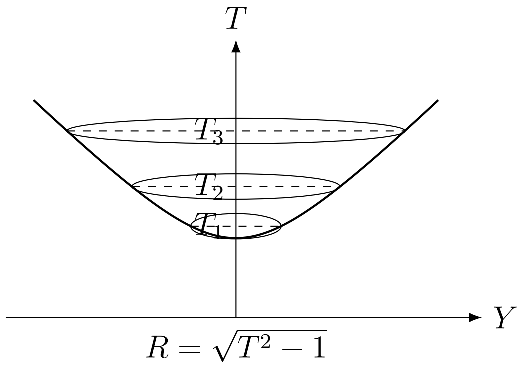


Figure 7: Horizontal time slices. Larger T gives a larger spatial slice $R = \sqrt{T^2 - 1}$ on the unit hyperboloid.

For two states, the invariant becomes

$$\chi_{12} = T_1 T_2 - R_1 R_2 (u_1 \cdot u_2). \tag{40}$$

After restoring units,

$$\langle Z_1, Z_2 \rangle_\eta = c^2 t_1 t_2 - x_1 \cdot x_2 = c^2 \tau_0^2 \chi_{12}. \tag{41}$$

The coordinate time t_i is therefore a frame reading, while χ_{12} is the invariant comparison between the embedded states.

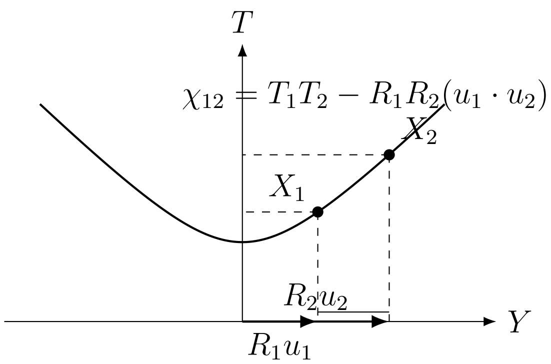


Figure 8: Two-state Lorentz product. The invariant combines the time product with the spatial alignment product.

6.1 Iterative Coordinate Time

For a recursive or discrete update, write

$$Y_k = R_k u_k, \qquad X_k = (T_k, Y_k), \qquad T_k^2 - Y_k \cdot Y_k = 1. \tag{42}$$

If the spatial embedding part updates by some rule

$$Y_{k+1} = F_k(Y_k), \tag{43}$$

then the future-sheet time coordinate is recovered algebraically:

$$T_{k+1} = \sqrt{1 + Y_{k+1} \cdot Y_{k+1}}. \tag{44}$$

The coordinate time at the next step is

$$t_{k+1} = \tau_0 T_{k+1}. \tag{45}$$

Hence the coordinate-time increment is

$$\Delta t_k = t_{k+1} - t_k = \tau_0 \left(T_{k+1} - T_k\right). \tag{46}$$

Equivalently,

$$\Delta t_k = \tau_0 \left(\sqrt{1 + Y_{k+1} \cdot Y_{k+1}} - \sqrt{1 + Y_k \cdot Y_k} \right). \tag{47}$$

The adjacent-step invariant is

$$\chi_{k,k+1} = T_k T_{k+1} - Y_k \cdot Y_{k+1}. \tag{48}$$

In separated direction form this is

$$\chi_{k,k+1} = T_k T_{k+1} - R_k R_{k+1}(u_k \cdot u_{k+1}). \tag{49}$$

So the recursive clock reading t_k is obtained from the zeroth coordinate, while the relation between successive embedded states is still governed by the same Lorentz inner product.

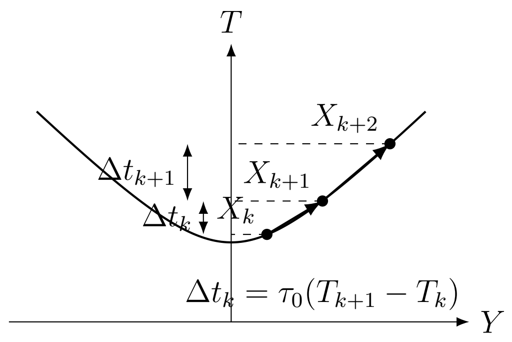


Figure 9: Iterative clock steps. A recursive sequence on the hyperboloid gives coordinate-time increments by projecting successive T-coordinates.

6.2 Differential Coordinate Time

For a smooth path, let

$$X(\lambda) = (T(\lambda), Y(\lambda)), \qquad T(\lambda)^2 - Y(\lambda) \cdot Y(\lambda) = 1. \tag{50}$$

Differentiating the constraint gives the tangent condition

$$T\dot{T} - Y \cdot \dot{Y} = 0, \tag{51}$$

where a dot means differentiation with respect to λ. Therefore

$$\dot{T} = \frac{Y \cdot \dot{Y}}{T}. \tag{52}$$

Since

$$t(\lambda) = \tau_0 T(\lambda), \tag{53}$$

the coordinate-time rate is

$$\dot{t} = \tau_0 \frac{Y \cdot \dot{Y}}{T}. \tag{54}$$

If $Y = Ru$, with $u \cdot u = 1$, then $u \cdot \dot{u} = 0$, and

$$Y \cdot \dot{Y} = R\dot{R}. \tag{55}$$

Thus

$$\dot{t} = \tau_0 \frac{R\dot{R}}{T}. \tag{56}$$

This means that, on this unit hyperboloid clock, a pure change of direction at fixed R changes the spatial direction and the pairwise invariant, but it does not by itself change the coordinate-time projection.

The induced positive hyperbolic speed along the path is obtained from the Lorentzian tangent norm:

$$v_H^2 = -\langle \dot{X}, \dot{X} \rangle_\eta = \dot{Y} \cdot \dot{Y} - \dot{T}^2. \tag{57}$$

In the separated form $Y = Ru$, this becomes

$$v_H^2 = \frac{\dot{R}^2}{T^2} + R^2 \dot{u} \cdot \dot{u}. \tag{58}$$

Again, no angular variable is required. Directional change is carried by $\dot{u} \cdot \dot{u}$, and coordinate-time change is carried by $\dot{T}$.

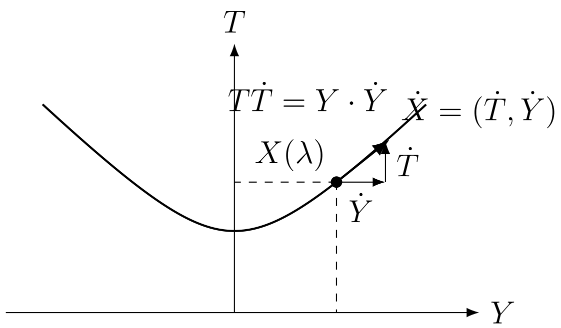


Figure 10: Tangent vector on the hyperboloid. The differential clock relation comes from differentiating the Lorentz constraint.

6.3 Plain-English Reading of Coordinate Time

The short version is that the same no-angle algebra works for time because time is not an angle either. It is the zeroth coordinate of the Lorentz vector. In the notation above,

$$X = (T, Y), \tag{59}$$

where T is the time-like coordinate and Y is the spatial part. The Lorentz constraint is

$$T^2 - Y \cdot Y = 1. \tag{60}$$

So once the spatial part Y is known, the future-sheet time coordinate is fixed algebraically:

$$T = \sqrt{1 + Y \cdot Y}. \tag{61}$$

After choosing the physical scale τ_0, coordinate time is

$$t = \tau_0 T. \tag{62}$$

If the spatial part is written as

$$Y = Ru, \qquad u \cdot u = 1, \tag{63}$$

then the constraint becomes

$$T^2 - R^2 = 1. \tag{64}$$

No rapidity angle is needed. The scalar R gives the spatial embedding magnitude, the vector u gives the spatial direction, and T is recovered from the Lorentz constraint.

For two states,

$$X_1 = (T_1, R_1 u_1), \qquad X_2 = (T_2, R_2 u_2), \tag{65}$$

the invariant is

$$\chi_{12} = T_1 T_2 - R_1 R_2 (u_1 \cdot u_2). \tag{66}$$

Thus the invariant comparison is made from a time part minus a spatial alignment part. The spatial alignment is still the dot product $u_1 \cdot u_2$, not an angle.

For an iterative update, if

$$Y_{k+1} = F_k(Y_k), \tag{67}$$

then the next coordinate-time reading is obtained by

$$T_{k+1} = \sqrt{1 + Y_{k+1} \cdot Y_{k+1}}, \qquad t_{k+1} = \tau_0 T_{k+1}. \tag{68}$$

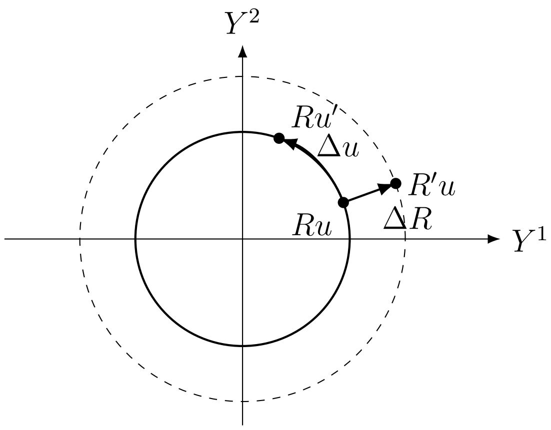
For a smooth path, differentiating $T^2 - Y \cdot Y = 1$ gives

$$\dot{T} = \frac{Y \cdot \dot{Y}}{T}, \qquad \dot{t} = \tau_0 \frac{Y \cdot \dot{Y}}{T}. \tag{69}$$

If $Y = Ru$, then $u \cdot \dot{u} = 0$, so

$$\dot{t} = \tau_0 \frac{R\dot{R}}{T}. \tag{70}$$

This shows the main intuition. Changing direction alone at fixed R changes the spatial orientation and pairwise invariants, but it does not by itself change the coordinate-time projection. Radial change changes T, and therefore changes the coordinate-time reading.



direction change at fixed R, radial change changes T

Figure 11: Radial change versus direction change. Changing u at fixed R moves round a slice, while changing R moves to another coordinate-time level.

In plain terms:

- direction is carried by u;
- spatial magnitude is carried by R;
- coordinate time is carried by T;
- invariant separation is carried by the Lorentz product;
- no angular variable or rapidity parameter has to be primitive.

7 Discussion

The construction separates three roles. First, the Lorentzian constraint fixes the embedded unit surface. Second, the unit direction vector records orientation in the spatial embedding sector. Third, the invariant χ_{12} compares two embedded states using only inner products.

This is not an approximation to an angular formula. It is the same directional information expressed without angular coordinates. In practical terms, the angle-free form is often cleaner because it generalises immediately from two spatial dimensions to n spatial dimensions, avoids coordinate singularities associated with angular charts, and keeps the comparison invariant under rotations of the embedded spatial coordinates.

The coordinate-time reading adds one caution. Coordinate time t is a projection in a chosen frame:

$$t = \tau_0 X^0. \tag{71}$$

The invariant object is not the coordinate time alone, but the Lorentzian relation between embedded vectors. In other words, time can be read algebraically from X^0, while separation is still read from $\langle X_1, X_2 \rangle_\eta$ or from the interval between physical spacetime vectors.

The notation α_i, ρ_i is used here only to avoid collision with survival weights, counts, or other model-specific symbols. Any application should choose notation that does not conflict with its own state variables.

8 From One Cycle to a Torus

The next construction is obtained by giving the state two independent closed directions. One direction moves round the main support cycle. The other direction moves round the local tube cycle. In ordinary angular coordinates this would normally be written as a torus parametrisation, but the same construction can be written without taking the angles as primitive. The topological background is the ordinary product $S^1 \times S^1$ [6].

Let

$$U = (U^1, U^2), \qquad U \cdot U = 1, \tag{72}$$

and

$$V = (V^1, V^2), \qquad V \cdot V = 1. \tag{73}$$

Here U is the main ring direction and V is the tube direction. Choose radii

$$R_0 > r_0 > 0. \tag{74}$$

The no-angle torus embedding is

$$\mathcal{T}(U, V) = \big((R_0 + r_0 V^1)U^1,\ (R_0 + r_0 V^1)U^2,\ r_0 V^2\big). \tag{75}$$

This is the same torus usually described by two angular variables, but here the primitive data are two unit direction pairs. The large cycle is carried by U, the tube cycle is carried by V, and the surface point is recovered algebraically from those two directions.

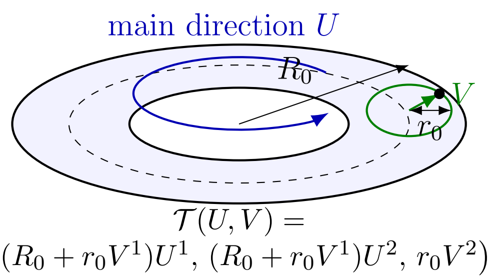


Figure 12: No-angle torus construction. The torus uses two unit directions: U for the main support cycle and V for the tube cycle.

In plain language, a circle is one closed update. A torus is two closed updates tied together. This matters for recursive histories because a rapidly updated object may not be perceived as the moving point sequence that generated it. If the update sequence closes, the observer may read the whole completed history as one extended support object.

9 Rational Closure on the Torus

The torus becomes a recursive object when the two directions update step by step. Write

$$U_{k+1} = A_k U_k, \qquad V_{k+1} = B_k V_k, \tag{76}$$

where the update matrices preserve unit length:

$$A_k^T A_k = I_2, \qquad B_k^T B_k = I_2. \tag{77}$$

The visible support point at step k is

$$P_k = \mathcal{T}(U_k, V_k). \tag{78}$$

A rational closure occurs when there is some $N > 0$ such that

$$U_N = U_0, \qquad V_N = V_0. \tag{79}$$

If the main cycle closes after p turns and the tube cycle closes after q turns, the closed history may be read as a (p, q)-type torus path. This is not a claim that the observer sees a literal rotating line. It says that if the observer resolves fewer frames than the update produces, the N-frame history can appear as one stable support trace.

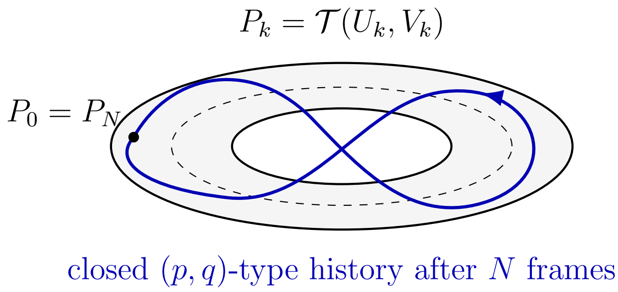


Figure 13: Rational closure on the torus. A recursive update may trace many frame-points, but if $U_N = U_0$ and $V_N = V_0$, the completed history closes.

This gives the natural bridge from sequential histories to apparent objects. The object is not merely one frame. It is the closed history read at a coarser perceptual or measurement rate. In that reading, N update frames are folded into one stable support.

10 Hopfion-Like Linked-Fibre Step

The torus supplies the two-cycle support. A hopfion-like structure requires one more ingredient: linked fibres. The minimal topological model is a map

$$h : S^3 \longrightarrow S^2. \tag{80}$$

The important feature is that the preimages of points in S^2 are closed loops in S^3, and distinct fibres are linked [7, 8]. This gives a stable class of closure, because a smooth deformation cannot unlink the fibres without changing the topological class.

For an algebraic version, write a point of $S^3 \subset \mathbb{C}^2$ as

$$(z_1, z_2) \in \mathbb{C}^2, \qquad |z_1|^2 + |z_2|^2 = 1. \tag{81}$$

Using unit direction pairs, set

$$z_1 = aU, \qquad z_2 = bV, \qquad a^2 + b^2 = 1, \tag{82}$$

where $U, V \in S^1$ are read as complex unit directions. For fixed a and b, the pair (U, V) describes a torus $S^1 \times S^1$ inside S^3. A standard Hopf-type projection is

$$h(z_1, z_2) = \big(2\,\mathrm{Re}(z_1\overline{z_2}), 2\,\mathrm{Im}(z_1\overline{z_2}), |z_1|^2 - |z_2|^2\big). \tag{83}$$

The topological check may be recorded by a class invariant Q_H. In a preserved hopfion-like sector,

$$Q_H(K_{n+1}) = Q_H(K_n). \tag{84}$$

This is the same cautious use as in the separate hopfion-like topology note: hopfions are witnesses of topological persistence, not foundations of the model [11].

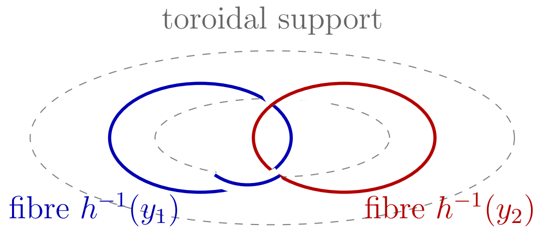


Figure 14: Hopfion-like linked fibres. The torus gives a two-cycle support, while the hopfion-like step adds linked closed fibres whose class cannot be removed by smooth deformation.

In plain terms, the construction moves in stages. A unit direction gives a clean phase or orientation state. A closed update of that direction gives a cycle. Two closed directions give a torus. Linked closed fibres on that toroidal support give the geometric skeleton of a hopfion-like support. The word “hopfion-like” is doing important work. A full physical hopfion would require a field, an energy functional, boundary conditions, and a topological charge computed inside that field theory. The present note only constructs the angle-free geometry needed to describe a linked-fibre support candidate.

11 Conclusion

Angular coordinates are optional. Directional separation can be represented by embedded unit direction vectors and Lorentzian inner products:

$$X_i = (\alpha_i, \rho_i u_i), \qquad \langle X_i, X_i \rangle_\eta = 1, \tag{85}$$

with pairwise invariant

$$\chi_{12} = \alpha_1\alpha_2 - \rho_1\rho_2(u_1 \cdot u_2). \tag{86}$$

In n spatial dimensions this becomes

$$\chi_{12} = \alpha_1\alpha_2 - \rho_1\rho_2 \sum_{a=1}^{n} u_1^a u_2^a. \tag{87}$$

The result is an angle-free, dimension-independent skeleton for directional comparison in a Lorentzian embedding. Coordinate time fits the same algebra: once a scale and frame are chosen, $t = \tau_0 X^0$, while iterative and differential time changes follow from the constraint $(X^0)^2 - X^{\text{space}} \cdot X^{\text{space}} = 1$. With two closed unit directions, the same method constructs a torus. With linked closed fibres, it gives the geometric skeleton of a hopfion-like support, while leaving any physical hopfion claim to a later field-theoretic model.

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