Correlations and differences between homogenous recurences and homogenous ODE

Let be $n \geqslant 1$a natural number, $a_n \neq 0, a_{n-1}, \ldots, a_1, a_0$ real numbers and consider the homogenous reccurence

$$a_n x_{k+n} + a_{n-1} x_{k+n-1} + \cdots + a_1 x_{k+1} + a_0 x_k = 0 \quad \forall k \in \mathbb{N}$$

and the homogenous Ordinary Differential Equation

$$a_n x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + \cdots + a_1 x'(t) + a_0 x(t) = 0 \quad \forall t \in I \subseteq \mathbb{R}$$

To each one it is attached the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

which can have real roots ρ of order ν and pairs of complex conjugate roots $r = \xi \pm i\psi$ of order ν. Until here, they look similar,but

For the ODE to each real root ρ of order ν we attach basis functions

$$e^{\rho t}, t e^{\rho t}, t^2 e^{\rho t}, \ldots t^{\nu-1} e^{\rho t},$$

in total ν functions and for each complex pair of roots $r = \xi \pm i\psi$ the basis functions

$$e^{\xi t}\cos(\psi t), e^{\xi t}\sin(\psi t), t e^{\xi t}\cos(\psi t), t e^{\xi t}\sin(\psi t), \ldots, t^{\nu-1} e^{\xi t}\cos(\psi t), t^{\nu-1} e^{\xi t}\sin(\psi t)$$

but for the reccurence to each real root of the characteristic eq we attach the basis sequences

$$\rho^n, n\rho^n, \ldots, n^{\nu-1}\rho^n$$

Sorry, I did not investigate what happens for pairs of complex roots like $r = \xi \pm i\psi$.

Now, for both, the general solution is a linear combination of basis functions. If we have an Initial Value Problem, then the values of constants can be found solving a linear algebraic system of n linear equations.

Example. For the reccurence let's take the classical Fibonnacci sequence

$$F_{k+2} - F_k - F_k = 0$$

here $n = 2; a_2 = 1, a_1 = -1, a_0 = -1$. The characteristic equation is

$$r^2 - r - 1 = 0$$

with roots $\rho_1 = \frac{1+\sqrt{5}}{2}$, of order $\nu_1 = 1$ and $\rho_2 = \frac{1-\sqrt{5}}{2}$ of order $\nu_2 = 1$. The basis functions will be ρ_1^n, ρ_2^n and the general solution will be

$$C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

and the constants are found from the system

$$\begin{cases} C_1\left(\dfrac{1+\sqrt{5}}{2}\right)+C_2\left(\dfrac{1-\sqrt{5}}{2}\right)=1 \\ C_1\left(\dfrac{1+\sqrt{5}}{2}\right)^2+C_2\left(\dfrac{1-\sqrt{5}}{2}\right)^2=1 \end{cases}$$

The "associated" ODE is

$$x''(t)-x'(t)-x(t)=0$$

the same characteristic eq. but with the general solution

$$x(t)=C_1e^{\rho_1 t}+C_2e^{\rho_2 t}$$

and, for the IVP, the constants are found solving the algebraic system

$$\begin{cases} C_1e^{\rho_1 t_0}+C_2e^{\rho_2 t_0}=x(t_0)\ \text{[given]} \\ C_1\rho_1e^{\rho_1 t_0}+C_2\rho_2e^{\rho_2 t_0}=x'(t_0)\ \text{[given]} \end{cases}$$