

A Trig-Free Recursive Reading of Beat Phenomena

Draft note for Jim Kelley

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1 Purpose

The usual textbook treatment of beat phenomena begins with the identity

$$\cos(2\pi f_1 t) + \cos(2\pi f_2 t) = 2 \cos(\pi(f_1 - f_2)t) \cos\left(2\pi \frac{f_1 + f_2}{2} t\right).$$

This identity is correct. The questionable step is pedagogical rather than algebraic. The product form may suggest that a new physical spectral component has appeared at

$$f_{\text{avg}} = \frac{f_1 + f_2}{2}.$$

However, the original Fourier content remains at f_1 and f_2 . The average-frequency term is a carrier-like factor in one time-domain factorisation, not necessarily a newly created spectral line.

The purpose of this note is to show the same beat mechanism without using sine or cosine as primary operators. The two tones are represented instead as recursive phase states, and the beat appears as their changing overlap.

2 Recursive phase state for one tone

A tone need not be introduced as a sine or cosine. It may be represented as a norm-preserving recursive phase state.

For tone i , define

$$\mathbf{u}_{i,n} = \begin{pmatrix} x_{i,n} \\ y_{i,n} \end{pmatrix}, \quad \mathbf{u}_{i,n+1} = R(a_i) \mathbf{u}_{i,n}.$$

Here n is the discrete time-step index, and $R(a_i)$ is the update matrix

$$R(a_i) = \frac{1}{1 + a_i^2} \begin{pmatrix} 1 - a_i^2 & -2a_i \\ 2a_i & 1 - a_i^2 \end{pmatrix}.$$

This is a rational, norm-preserving rotation update. It does not require sine or cosine as primitive functions.

Indeed,

$$R(a_i)^T R(a_i) = I,$$

and therefore

$$x_{i,n+1}^2 + y_{i,n+1}^2 = x_{i,n}^2 + y_{i,n}^2.$$

Thus the phase state remains on the same circle:

$$\|\mathbf{u}_{i,n+1}\|^2 = \|\mathbf{u}_{i,n}\|^2.$$

The observed scalar signal associated with this tone may be taken as the projection onto the x -axis:

$$p_{i,n} = A_i x_{i,n}.$$

So the “wave” is generated by recursive norm-preserving phase motion, not by calling sin or cos as primary operators.

3 Two tones and the pressure sum

For two tones, the observed pressure signal is simply the projected sum

$$p_n = A_1 x_{1,n} + A_2 x_{2,n}.$$

The two generators remain distinct:

$$\mathbf{u}_{1,n+1} = R(a_1)\mathbf{u}_{1,n}, \quad \mathbf{u}_{2,n+1} = R(a_2)\mathbf{u}_{2,n}.$$

The beat is not produced by introducing an average-frequency oscillator. It appears in the changing overlap between the two recursive phase states.

Define the combined phase vector

$$\mathbf{U}_n = A_1 \mathbf{u}_{1,n} + A_2 \mathbf{u}_{2,n}.$$

Then the squared beat envelope is

$$B_n^2 = \|A_1 \mathbf{u}_{1,n} + A_2 \mathbf{u}_{2,n}\|^2.$$

Expanding the norm gives

$$B_n^2 = A_1^2 \|\mathbf{u}_{1,n}\|^2 + A_2^2 \|\mathbf{u}_{2,n}\|^2 + 2A_1 A_2 \mathbf{u}_{1,n} \cdot \mathbf{u}_{2,n}.$$

If each phase state is normalised so that

$$\|\mathbf{u}_{1,n}\| = \|\mathbf{u}_{2,n}\| = 1,$$

then

$$B_n^2 = A_1^2 + A_2^2 + 2A_1 A_2 (x_{1,n} x_{2,n} + y_{1,n} y_{2,n}).$$

The final term,

$$\mathbf{u}_{1,n} \cdot \mathbf{u}_{2,n} = x_{1,n} x_{2,n} + y_{1,n} y_{2,n},$$

is the beat mechanism. It changes slowly when the two phase updates are close. No average-frequency oscillator has been introduced.

4 Relative update and beat rate

The beat is governed by the relative phase update between the two tones.

Define

$$Q = R(a_1)^{-1} R(a_2).$$

Since the updates are norm-preserving, this relative update also preserves norm. The relative overlap may be written as

$$m_n = \mathbf{u}_{1,n} \cdot \mathbf{u}_{2,n}.$$

When the two update matrices commute, as they do for planar rotations of this form,

$$m_n = \mathbf{u}_{1,0} \cdot Q^n \mathbf{u}_{2,0}.$$

Therefore

$$B_n^2 = A_1^2 + A_2^2 + 2A_1A_2m_n.$$

The audible beat is governed by the closure cycle of Q , not by a newly created average-frequency component.

Let N_b be the smallest step count such that

$$Q^{N_b} \approx I.$$

Then, for time step Δt , the audible beat rate is

$$f_{\text{beat}} = \frac{1}{N_b \Delta t}.$$

This expresses the beat as a relative closure rate.

5 The factor-of-two trap

The usual product identity introduces a signed modulation factor,

$$\cos(\pi(f_1 - f_2)t).$$

If written in the form

$$2 \cos(2\pi f_{\text{mod}}t) \cos(2\pi f_{\text{avg}}t),$$

then

$$f_{\text{avg}} = \frac{f_1 + f_2}{2}, \quad f_{\text{mod}} = \frac{|f_1 - f_2|}{2}.$$

But the audible beat rate, counted as loudness maxima, is

$$f_{\text{beat}} = |f_1 - f_2|.$$

Thus

$$f_{\text{beat}} = 2f_{\text{mod}}.$$

In recursive language this becomes clear.

The relative phase overlap closes after N_b steps:

$$Q^{N_b} \approx I.$$

That gives the audible beat rate

$$f_{\text{beat}} = \frac{1}{N_b \Delta t}.$$

The signed algebraic envelope from the product identity returns to the same signed state only after twice that interval:

$$N_{\text{signed}} = 2N_b.$$

Therefore

$$f_{\text{mod}} = \frac{1}{N_{\text{signed}}\Delta t} = \frac{1}{2N_b\Delta t} = \frac{f_{\text{beat}}}{2}.$$

So the clean distinction is

$$f_{\text{beat}} = \text{relative closure rate of } Q,$$

while

$$f_{\text{mod}} = \text{signed-envelope convention rate.}$$

The factor of two is a bookkeeping effect introduced by the signed product representation.

6 Long chain of tones

For a chain of tones, define a phase state for each component:

$$\mathbf{u}_{i,n+1} = R(a_i)\mathbf{u}_{i,n}.$$

The observed pressure signal is

$$p_n = \sum_i A_i x_{i,n}.$$

The total recursive phase vector is

$$\mathbf{U}_n = \sum_i A_i \mathbf{u}_{i,n}.$$

The squared envelope is then

$$B_n^2 = \left\| \sum_i A_i \mathbf{u}_{i,n} \right\|^2.$$

Expanding gives

$$B_n^2 = \sum_i A_i^2 \|\mathbf{u}_{i,n}\|^2 + 2 \sum_{i<j} A_i A_j \mathbf{u}_{i,n} \cdot \mathbf{u}_{j,n}.$$

For normalised phase states,

$$\|\mathbf{u}_{i,n}\| = 1,$$

this reduces to

$$B_n^2 = \sum_i A_i^2 + 2 \sum_{i<j} A_i A_j \mathbf{u}_{i,n} \cdot \mathbf{u}_{j,n}.$$

So a long chain of tones becomes

individual recursive phase states \rightarrow projected pressure sum \rightarrow pairwise phase overlaps \rightarrow audible beat structure

The beat structure is therefore built from pairwise relative closures, not from new average-frequency spectral components.

7 Comparison with the textbook identity

The textbook identity remains valid:

$$\cos(2\pi f_1 t) + \cos(2\pi f_2 t) = 2 \cos(\pi(f_1 - f_2)t) \cos\left(2\pi \frac{f_1 + f_2}{2} t\right).$$

But the recursive formulation makes the interpretation clearer.
The original generators remain two tones:

$$f_1, \quad f_2.$$

The average term,

$$\frac{f_1 + f_2}{2},$$

belongs to one algebraic factorisation of the time-domain sum. It is not automatically a new Fourier component.

A defensible statement is therefore:

The algebraic product form is valid, but the common pedagogical interpretation is misleading when it treats the average-frequency term as a physically present spectral component.

The recursive form keeps the original tones as the generators and makes the beat appear as their changing overlap.

8 Compact conclusion

Beat phenomena can be described without treating sin or cos as primary operators.

Each tone is a norm-preserving recursive phase state:

$$\mathbf{u}_{i,n+1} = R(a_i)\mathbf{u}_{i,n}.$$

The observed sound is a projection:

$$p_n = \sum_i A_i x_{i,n}.$$

The beat structure comes from pairwise overlaps:

$$\mathbf{u}_{i,n} \cdot \mathbf{u}_{j,n}.$$

The audible beat rate is the relative closure rate of the phase updates:

$$f_{\text{beat}} = \frac{1}{N_b \Delta t}, \quad Q^{N_b} \approx I.$$

The half-rate modulation term in the usual identity is a signed-envelope convention:

$$f_{\text{mod}} = \frac{1}{2} f_{\text{beat}}.$$

Thus the recursive formulation supports the central point: the average-frequency term is a representation inside the algebraic rewrite, not a newly created spectral line.