An Angle-Free Note on the Repeated Appearance of 2/3: Koide Projectors, $D = 3 + \epsilon$ Deformations, FFGFT Inputs, and Recursive Closure

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May 2026

Abstract

The number 2/3 appears in several different settings: the charged-lepton Koide relation, complement laws of the form $1 - 1/D$, rank splits such as 16/24, and recursive closure rules. This note gives an angle-free algebraic comparison. The main diagnostic is a formal deformation $D = 3 + \epsilon$. At $D = 3$ several routes meet at 2/3, but under deformation they separate. This makes the ϵ-response a useful test of which mechanism is actually being claimed. The FFGFT case is treated as a mass-generation route near the Koide projector surface, not as a claimed derivation of exact Koide closure.

1 Purpose and Scope

This note does not try to prove that the repeated occurrence of 2/3 has a single origin. It does the opposite: it separates several distinct algebraic routes that all meet at 2/3 when $D = 3$.

The central principle is:

> At $D = 3$, several mechanisms can produce 2/3. At $D = 3 + \epsilon$, they usually separate. The deformation tells us which road we were actually on.

No trigonometric parametrisation is used. The Koide relation is written using only vectors, inner products, projectors, ranks, and quadratic weights.

2 Notation Key

This section fixes the main symbols. The note uses no trigonometric parametrisation; all statements are written in terms of vectors, projectors, ranks, ratios, and recursive closure rules.

Symbol	Meaning
m_i	A positive mass in a finite list. In the charged-lepton case, $i \in \{e, \mu, \tau\}$.
m_e, m_μ, m_τ	Electron, muon, and tau masses. These are inserted only when a numerical Koide value is computed.
x	Square-root mass vector, with components $x_i = \sqrt{m_i}$. For charged leptons, $x = (\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})^T$.
u	Democratic three-component vector, $u = (1, 1, 1)^T$.
u_D	Democratic D-component vector, $u_D = (1, 1, \ldots, 1)^T \in \mathbb{R}^D$.
I	Identity matrix or identity operator on the space being discussed.

Symbol	Meaning
P	Democratic projector in the three-component case, $P = uu^T/(u^Tu) = uu^T/3$.
$P_\perp$	Residual projector, $P_\perp = I - P$. It projects onto the part left after the democratic component is removed.
P_D	Democratic projector in D components, $P_D = u_D u_D^T/D$.
$P_{D,\perp}$	D-component residual projector, $P_{D,\perp} = I - P_D$.
A, B	Quadratic weights in the three-component projector split: $A = x^T Px$ and $B = x^T P_\perp x$. Koide equality is $A = B$.
A_D, B_D	D-component weights: $A_D = x^T P_D x$ and $B_D = x^T P_{D,\perp} x$.
Q	Koide scalar in three components, $Q = x^T x/(u^T x)^2$. For charged leptons this is the usual Koide quantity.
Q_D	D-component Koide-like scalar, $Q_D = x^T x/(u_D^T x)^2$.
Δ_K	Koide diagnostic, $\Delta_K = 3Q - 2 = B/A - 1$. It vanishes exactly when $A = B$.
D	Number of components in the algebraic continuation, or the topological spatial dimension when discussing an exponent step. These uses are kept explicit in the text.
ϵ	Formal deformation parameter defined by $D = 3 + \epsilon$. It is a diagnostic deformation, not a claim that physical space has non-integer component count.
$Q_{\text{proj}}(D)$	Equal-projector continuation, $Q_{\text{proj}}(D) = 2/D$.
$p_{\text{comp}}(D)$	Complement-of-step route, $p_{\text{comp}}(D) = 1 - 1/D$.
$Q_{\text{mid}}(D)$	Midpoint route, $Q_{\text{mid}}(D) = \frac{1}{2}(1 + 1/D)$.
M, R	Short-hands used in the uncertainty calculation: $M = \sum_i m_i$ and $R = \sum_i \sqrt{m_i}$.
σ_Q	Propagated one-sigma uncertainty in Q from the charged-lepton mass uncertainties.
σ_{Δ_K}	Propagated one-sigma uncertainty in Δ_K; since $\Delta_K = 3Q - 2$, one has $\sigma_{\Delta_K} = 3\sigma_Q$.
ξ	FFGFT/T0 dimensionless parameter quoted publicly as $\xi = 4/30000$.
D_f	Claimed FFGFT fractal dimension, $D_f = 3 - \xi$. This is not the same symbol as the topological dimension D.
r_i, p_i	FFGFT mass-generator inputs: r_i are rational coefficients and p_i are rational exponents. These are unrelated to the scalar recursive ratio r.
v	Common electroweak scale factor in the quoted FFGFT mass formula $m_i = r_i \xi^{p_i} v$. It cancels from Q.
Q_{FFGFT}	Koide scalar generated from the quoted FFGFT inputs after forming $x_i = \sqrt{r_i}\xi^{p_i/2}$.
V, J, P_+, P_-	Rank-split mediation notation: V is a finite mediation space, J is an involution with $J^2 = I$, and $P_\pm = (I \pm J)/2$ are the associated projectors.
S, L, r	Recursive closure notation: S is the closed recursive whole, L is the target whole, and r is the unresolved recursive remainder ratio.
r_n, R_n, p_n	Variable recursive closure notation: r_n is the step-dependent unresolved ratio, R_n is the remaining product after n steps, and $p_n = L(1 - r_n)R_n$ is the resolved piece at step n.

3 Koide as an Angle-Free Projector Statement

For three positive charged-lepton masses, define the square-root mass vector

$$x = \begin{pmatrix} \sqrt{m_e} \\ \sqrt{m_\mu} \\ \sqrt{m_\tau} \end{pmatrix}, \qquad u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The Koide quantity is

$$Q = \frac{m_e + m_\mu + m_\tau}{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2}.$$

Since $x_i = \sqrt{m_i}$, this becomes

$$Q = \frac{x^T x}{(u^T x)^2}.$$

Define the democratic projector

$$P = \frac{uu^T}{u^T u}.$$

For three components, $u^T u = 3$, so

$$P = \frac{uu^T}{3}.$$

Define the residual projector

$$P_\perp = I - P.$$

Now define the democratic and residual quadratic weights

$$A = x^T P x, \qquad B = x^T P_\perp x.$$

Because

$$P + P_\perp = I,$$

we have

$$A + B = x^T (P + P_\perp) x = x^T x.$$

Also,

$$\begin{aligned} A &= x^T \left(\frac{uu^T}{3} \right) x \\ &= \frac{x^T u u^T x}{3} \\ &= \frac{(u^T x)^2}{3}. \end{aligned}$$

Therefore

$$(u^T x)^2 = 3A.$$

Substituting these two identities into the Koide expression gives

$$\begin{aligned} Q &= \frac{x^T x}{(u^T x)^2} \\ &= \frac{A + B}{3A}. \end{aligned}$$

Thus the Koide value $Q = 2/3$ is equivalent to

$$\begin{aligned} \frac{A+B}{3A} &= \frac{2}{3}, \\ A + B &= 2A, \\ B &= A. \end{aligned}$$

So, in this angle-free form, the Koide relation says:

The square-root charged-lepton mass vector has equal quadratic weight in the democratic projector sector and the residual projector sector.

4 The D-Component Projector Form

For an integer number D of components, define

$$u_D = (1, 1, \ldots, 1)^T \in \mathbb{R}^D, \qquad P_D = \frac{u_D u_D^T}{u_D^T u_D} = \frac{u_D u_D^T}{D},$$

and

$$P_{D,\perp} = I - P_D.$$

Let

$$A_D = x^T P_D x, \qquad B_D = x^T P_{D,\perp} x.$$

Then

$$A_D + B_D = x^T x,$$

and

$$\begin{aligned} A_D &= x^T \left(\frac{u_D u_D^T}{D} \right) x \\ &= \frac{(u_D^T x)^2}{D}. \end{aligned}$$

Therefore

$$(u_D^T x)^2 = D A_D.$$

The D-component Koide-like quantity is

$$Q_D = \frac{x^T x}{(u_D^T x)^2}.$$

Substitution gives

$$Q_D = \frac{A_D + B_D}{D A_D}.$$

Therefore

$$Q_D = \frac{2}{D}$$

if and only if

$$\begin{aligned} \frac{A_D + B_D}{D A_D} &= \frac{2}{D}, \\ A_D + B_D &= 2A_D, \\ B_D &= A_D. \end{aligned}$$

Thus the equal-projector continuation gives

$$Q_{\text{proj}}(D) = \frac{2}{D}.$$

The expression $D = 3 + \epsilon$ is then understood as a formal analytic continuation of this scalar function, not as a literal vector space of non-integer dimension.

5 Three Routes That Meet at $D = 3$

There are at least three simple algebraic routes to 2/3 at $D = 3$.

5.1 Route 1: Equal Projector Weight

Preserving the condition $A_D = B_D$ gives

$$Q_{\mathrm{proj}}(D) = \frac{2}{D}.$$

At $D = 3$,

$$Q_{\mathrm{proj}}(3) = \frac{2}{3}.$$

5.2 Route 2: Complement of a Dimensional Step

If a primitive step is

$$\Delta p(D) = \frac{1}{D},$$

then its complement is

$$p_{\mathrm{comp}}(D) = 1 - \frac{1}{D}.$$

At $D = 3$,

$$p_{\mathrm{comp}}(3) = 1 - \frac{1}{3} = \frac{2}{3}.$$

5.3 Route 3: Midpoint of the Allowed Koide Interval

For positive x_i,

$$\frac{1}{D} \le Q_D \le 1.$$

The lower bound follows from Cauchy's inequality:

$$(u_D^T x)^2 \le (u_D^T u_D)(x^T x) = D x^T x,$$

so

$$Q_D = \frac{x^T x}{(u_D^T x)^2} \ge \frac{1}{D}.$$

The upper bound follows because, for positive x_i,

$$(u_D^T x)^2 = \left(\sum_i x_i\right)^2 = \sum_i x_i^2 + 2\sum_{i<j} x_i x_j \ge \sum_i x_i^2 = x^T x.$$

Hence

$$Q_D \le 1.$$

The midpoint of the interval $[1/D, 1]$ is

$$Q_{\mathrm{mid}}(D) = \frac{1}{2}\left(1 + \frac{1}{D}\right).$$

At $D = 3$,

$$Q_{\mathrm{mid}}(3) = \frac{1}{2}\left(1 + \frac{1}{3}\right) = \frac{2}{3}.$$

6 Symbolic ϵ Expansions

Set

$$D = 3 + \epsilon.$$

The three routes have the following Taylor expansions through order ϵ^2:

$$\begin{aligned}
Q_{\mathrm{proj}}(3+\epsilon) &= \frac{2}{3+\epsilon} \\
&= \frac{2}{3} - \frac{2}{9}\epsilon + \frac{2}{27}\epsilon^2 + O(\epsilon^3), \\
p_{\mathrm{comp}}(3+\epsilon) &= 1 - \frac{1}{3+\epsilon} \\
&= \frac{2}{3} + \frac{1}{9}\epsilon - \frac{1}{27}\epsilon^2 + O(\epsilon^3), \\
Q_{\mathrm{mid}}(3+\epsilon) &= \frac{1}{2}\left(1 + \frac{1}{3+\epsilon}\right) \\
&= \frac{2}{3} - \frac{1}{18}\epsilon + \frac{1}{54}\epsilon^2 + O(\epsilon^3).
\end{aligned}$$

The first derivatives at $\epsilon = 0$ are therefore

$$Q'_{\mathrm{proj}}(0) = -\frac{2}{9}, \qquad p'_{\mathrm{comp}}(0) = \frac{1}{9}, \qquad Q'_{\mathrm{mid}}(0) = -\frac{1}{18}.$$

The second derivatives at $\epsilon = 0$ are

$$Q''_{\mathrm{proj}}(0) = \frac{4}{27}, \qquad p''_{\mathrm{comp}}(0) = -\frac{2}{27}, \qquad Q''_{\mathrm{mid}}(0) = \frac{1}{27}.$$

This is the main discriminator: the three routes agree at $D = 3$, but their first-order responses differ in both sign and magnitude.

7 The Algebraic Koide Diagnostic

For the three-component charged-lepton case define

$$\Delta_K = 3Q - 2.$$

Using

$$Q = \frac{A+B}{3A},$$

we get

$$\begin{aligned}
\Delta_K &= 3\left(\frac{A+B}{3A}\right) - 2 \\
&= \frac{A+B}{A} - 2 \\
&= 1 + \frac{B}{A} - 2 \\
&= \frac{B}{A} - 1.
\end{aligned}$$

Thus

$$\Delta_K = 0 \quad \Longleftrightarrow \quad \frac{B}{A} = 1 \quad \Longleftrightarrow \quad A = B.$$

For the D-component formalisation the analogous diagnostic is

$$\Delta_K^{(D)} = DQ_D - 2 = \frac{B_D}{A_D} - 1.$$

This is useful because any proposed mass generator can be checked by generating the vector x, computing A and B, and then measuring the residual $B/A - 1$.

8 Worked Numerical Example: Physical Charged Leptons

Using the charged-lepton masses

$$m_e = 0.51099895000\,\mathrm{MeV}, \quad m_\mu = 105.6583755\,\mathrm{MeV}, \quad m_\tau = 1776.93\,\mathrm{MeV},$$

the square-root mass vector is approximately

$$x \approx (0.7148419056,\ 10.2790259996,\ 42.1536475290)^T.$$

For this vector,

$$A \approx 941.552798942, \qquad B \approx 941.546575508.$$

Therefore

$$Q \approx 0.6666644634, \qquad \Delta_K = 3Q - 2 \approx -6.6097565 \times 10^{-6}.$$

Equivalently,

$$\frac{B}{A} - 1 \approx -6.6097565 \times 10^{-6}.$$

The physical charged-lepton vector is therefore extremely close to equal projector weight, but not exactly equal when current central values are inserted.

9 Worked Numerical Example: Small ϵ Separation

For a small formal deformation, take $\epsilon = 0.01$, so $D = 3.01$. Then

Route	Formula	Value at $\epsilon = 0.01$
Equal projector	$2/(3+\epsilon)$	0.6644518272
Complement of step	$1 - 1/(3+\epsilon)$	0.6677740864
Midpoint	$\frac{1}{2}(1 + 1/(3+\epsilon))$	0.6661129568

All three routes pass through 2/3 at $\epsilon = 0$, but they immediately diverge once the dimension is deformed.

10 Recursive Closure and the Same Complement

The Zeno-style recursive closure equation is

$$S = (1-r)L + rS,$$

where L is the target whole, r is the unresolved recursive remainder, and $1 - r$ is the resolved complement. Then

$$\begin{aligned} S - rS &= (1-r)L, \\ S(1-r) &= L(1-r), \\ S &= L, \end{aligned}$$

provided $r \neq 1$.

For $r = 1/2$,

$$S = \frac{1}{2}L + \frac{1}{2}S,$$

so

$$\frac{L}{2} + \frac{L}{4} + \frac{L}{8} + \cdots = L.$$

For $r = 1/3$,

$$S = \frac{2}{3}L + \frac{1}{3}S,$$

so

$$\frac{2L}{3} + \frac{2L}{9} + \frac{2L}{27} + \cdots = L.$$

For $r = 2/3$,

$$S = \frac{1}{3}L + \frac{2}{3}S,$$

so

$$\frac{L}{3} + \frac{2L}{9} + \frac{4L}{27} + \cdots = L.$$

Thus, if the unresolved recursive copy is 1/3 of the whole, the resolved closure complement is 2/3 of the whole. In this setting, 2/3 is not inserted as a coincidence; it is the algebraic complement required to close a 1/3 recursive remainder.

For a variable recursive sequence, let r_n be the unresolved fraction at step n. Define

$$R_0 = 1, \qquad R_{n+1} = R_n r_n.$$

The resolved piece at step n is

$$p_n = L(1 - r_n)R_n.$$

Then the partial sum telescopes:

$$\begin{aligned}\sum_{n=0}^{N} p_n &= L\sum_{n=0}^{N}(1 - r_n)R_n \\ &= L\sum_{n=0}^{N}(R_n - R_{n+1}) \\ &= L(1 - R_{N+1}).\end{aligned}$$

If

$$R_N \longrightarrow 0,$$

then

$$\sum_{n=0}^{\infty} p_n = L.$$

This closes the recursive sequence without requiring a final step after infinitely many steps. The unresolved remainder tends to zero.

A caveat is needed. Recursive spatial closure and measured curve length are not always the same. A fractal can be spatially bounded while its arc length diverges if the length measure expands under recursion. Therefore the safe statement is:

> A recursive construction closes in the chosen measure when the residual product tends to zero. If the chosen measure is arc length and the length-scaling factor exceeds one, the arc length may diverge even while the spatial support remains bounded.

11 Rank-Split Algebra for a $1/3 + 2/3$ Mediation Frame

Suppose a finite mediation space V has

$$\dim(V) = 24.$$

Let $J : V \to V$ be an involution:

$$J^2 = I.$$

The associated projectors are

$$P_+ = \frac{I+J}{2}, \qquad P_- = \frac{I-J}{2}.$$

They satisfy

$$P_+^2 = P_+, \qquad P_-^2 = P_-, \qquad P_+ + P_- = I, \qquad P_+P_- = 0.$$

If the ranks are

$$\operatorname{rank}(P_+) = 8, \qquad \operatorname{rank}(P_-) = 16,$$

then

$$\frac{\operatorname{rank}(P_+)}{\dim(V)} = \frac{8}{24} = \frac{1}{3}, \qquad \frac{\operatorname{rank}(P_-)}{\dim(V)} = \frac{16}{24} = \frac{2}{3}.$$

This makes the ratio an algebraic rank statement, not merely a repeated numerical observation. The ratio is meaningful only after the object, involution, projectors, ranks, and invariants are all specified.

12 FFGFT / T0 Inputs Used Here

Johann Pascher clarified that the FFGFT/T0 charged-lepton inputs are exact rational inputs in the public construction, not rounded fitting parameters. The values used here are therefore:

$$\begin{aligned} \xi_0 &= \frac{4}{30000} = \frac{1}{7500}, \\ r_e &= \frac{4}{3}, \quad p_e = \frac{3}{2}, \\ r_\mu &= \frac{16}{5}, \quad p_\mu = 1, \\ r_\tau &= \frac{25}{9}, \quad p_\tau = \frac{2}{3}. \end{aligned}$$

The mass generator is

$$m_i = r_i \xi_0^{p_i} v,$$

with $v = 246\,\mathrm{GeV} = 246000\,\mathrm{MeV}$ when explicit mass units are required. The scale v cancels from the Koide diagnostic, so the projector calculation depends only on r_i, p_i, ξ_0.

This note therefore treats the FFGFT charged-lepton output as a fixed geometric mass-generation route. It does not treat the Koide residual as a rounding error.

From

$$m_i = r_i \xi^{p_i} v,$$

one obtains

$$x_i = \sqrt{m_i} = \sqrt{v}\,\sqrt{r_i}\,\xi^{p_i/2}.$$

Since the common factor $\sqrt{v}$ cancels from Q, the FFGFT-generated Koide quantity is

$$Q_{\text{FFGFT}} = \frac{\sum_i r_i \xi^{p_i}}{\left(\sum_i \sqrt{r_i}\,\xi^{p_i/2}\right)^2}.$$

The exact Koide target is therefore

$$3\sum_i r_i \xi^{p_i} = 2\left(\sum_i \sqrt{r_i}\,\xi^{p_i/2}\right)^2.$$

Equivalently, after generating x from the FFGFT inputs, one must obtain

$$A = B,$$

where

$$A = x^T P x, \qquad B = x^T P_\perp x.$$

Input	Formula or values used	Status in this note
Geometric parameter	$\xi = 4/30000$	Taken as quoted input
Fractal dimension	$D_f = 3 - \xi$	Used only in the fractal-step test
Mass generator	$m_i = r_i \xi^{p_i} v$	Used algebraically; v cancels from Koide
Charged-lepton coefficients	$r_e = 4/3$, $r_\mu = 16/5$, $r_\tau = 25/9$	Taken from public summary
Charged-lepton exponents	$p_e = 3/2$, $p_\mu = 1$, $p_\tau = 2/3$	Taken from public summary
Step claim	$\Delta p = 1/3 = 1/D$ with $D = 3$	Treated as topological-dimensional unless stated otherwise

Table 2: Public FFGFT/T0 assumptions used in this note.

13 FFGFT Numerical Diagnostic from the Quoted Inputs

Using only the quoted public charged-lepton inputs

$$\xi = \frac{4}{30000}, \quad (r_e, r_\mu, r_\tau) = \left(\frac{4}{3}, \frac{16}{5}, \frac{25}{9}\right), \quad (p_e, p_\mu, p_\tau) = \left(\frac{3}{2}, 1, \frac{2}{3}\right),$$

the scale-free square-root vector is

$$x_i = \sqrt{r_i}\, \xi^{p_i/2}.$$

Numerically,

$$x \approx (0.0014327599,\ 0.0206559112,\ 0.0851454925)^T.$$

This gives

$$A \approx 0.00383305528, \qquad B \approx 0.00384541908,$$

and hence

$$Q_{\mathrm{FFGFT}} \approx 0.6677418577, \qquad \Delta_K = 3Q_{\mathrm{FFGFT}} - 2 \approx 0.0032255731.$$

Equivalently,

$$\frac{B}{A} - 1 \approx 0.0032255731.$$

Johann Pascher has clarified that this is the intended bare output of the exact rational FFGFT inputs. Thus

$$Q_{\mathrm{FFGFT}} \approx 0.6677418577$$

is not a failure to derive Koide, because FFGFT does not claim to derive exact Koide closure. It is better described as a Koide-proximity diagnostic: the FFGFT-generated vector lies close to the equal-projector surface $A = B$, but is not exactly on it.

The residual

$$Q_{\mathrm{FFGFT}} - \frac{2}{3} \approx 0.0010751910$$

is therefore a structural by-product of the quoted torus-mode coefficients and exponents, not a hidden rounding artefact.

14 Cross-Framework Residual Check

Paul Phillips observed that the FFGFT Koide residual is numerically close to a simple expression involving a convexity parameter S. If

$$S(1-S)=\frac{1}{20},$$

then the large-root solution is

$$S=\frac{1+\sqrt{4/5}}{2}\approx 0.9472135955.$$

Hence

$$1-S\approx 0.0527864045,$$

and

$$\frac{1-S}{49}\approx 0.0010772736.$$

The FFGFT residual is

$$Q_{\text{FFGFT}}-\frac{2}{3}\approx 0.0010751910.$$

The difference is therefore

$$\left(Q_{\text{FFGFT}}-\frac{2}{3}\right)-\frac{1-S}{49}\approx -2.08\times 10^{-6}.$$

So the match is close, but not exact. It is best treated as a candidate cross-framework deformation term, not as an established identity.

15 Sensitivity of the FFGFT Diagnostic to ξ

For the quoted FFGFT inputs, numerical differentiation gives approximately

$$\left.\frac{\partial Q_{\text{FFGFT}}}{\partial \xi}\right|_{\xi=4/30000}\approx -2.83443\times 10^{2},$$

and therefore

$$\left.\frac{\partial Q_{\text{FFGFT}}}{\partial \ln \xi}\right|_{\xi=4/30000}=\xi\frac{\partial Q_{\text{FFGFT}}}{\partial \xi}\approx -3.77924\times 10^{-2}.$$

A small fractional shift in ξ therefore produces the approximate response

$$\delta Q_{\text{FFGFT}}\approx -3.77924\times 10^{-2}\,\delta(\ln \xi).$$

For example:

Change in ξ	Q_{FFGFT}	Shift from base value
−0.1%	0.6677796679	$+3.7810\times 10^{-5}$
+0.1%	0.6677040831	-3.7775×10^{-5}
−1.0%	0.6681215730	$+3.7972\times 10^{-4}$
+1.0%	0.6673657020	-3.7616×10^{-4}

This sensitivity is useful because it tells us whether a claimed fractal correction is numerically relevant compared with the desired Koide precision.

16 Experimental Scale of a Detectable Shift in Q

It is useful to put the deformation scale beside the present charged-lepton uncertainties. Write

$$M = \sum_i m_i, \qquad R = \sum_i \sqrt{m_i}, \qquad Q = \frac{M}{R^2}.$$

Then

$$\frac{\partial Q}{\partial m_i} = \frac{1}{R^2} - \frac{M}{R^3\sqrt{m_i}}.$$

Using the PDG 2024 one-sigma mass uncertainties, approximately

$$\sigma_{m_e} \simeq 1.5 \times 10^{-10}\,\text{MeV}, \quad \sigma_{m_\mu} \simeq 2.3 \times 10^{-6}\,\text{MeV}, \quad \sigma_{m_\tau} \simeq 0.09\,\text{MeV},$$

and treating them as independent, linear propagation gives

$$\sigma_Q^2 = \sum_i \left(\frac{\partial Q}{\partial m_i}\sigma_{m_i}\right)^2, \qquad \sigma_Q \approx 5.08 \times 10^{-6}.$$

Equivalently,

$$\sigma_{\Delta_K} = 3\sigma_Q \approx 1.52 \times 10^{-5}.$$

The uncertainty is dominated by the tau mass. This gives the practical scale of the $D = 3 + \epsilon$ discriminator: an equal-projector deformation needs only

$$|\epsilon| \approx \frac{\sigma_Q}{2/9} \approx 2.29 \times 10^{-5}$$

to move Q by one present experimental sigma. The complement route needs

$$|\epsilon| \approx \frac{\sigma_Q}{1/9} \approx 4.57 \times 10^{-5},$$

and the midpoint route needs

$$|\epsilon| \approx \frac{\sigma_Q}{1/18} \approx 9.15 \times 10^{-5}.$$

For the quoted FFGFT hybrid rung, where $dQ/d\epsilon \approx -0.0994008$, the corresponding scale is

$$|\epsilon| \approx 5.11 \times 10^{-5}.$$

For the quoted FFGFT ξ-dependence, using

$$\frac{\partial Q}{\partial \ln \xi} \approx -3.77924 \times 10^{-2},$$

a fractional parameter shift

$$|\delta \ln \xi| \approx 1.34 \times 10^{-4},$$

that is, about 0.0134%, changes Q by one present experimental sigma. For one-at-a-time coefficient changes in the quoted FFGFT vector, the same one-sigma shift is produced by fractional changes of approximately

$$|\delta \ln r_e| \approx 5.81 \times 10^{-4}, \qquad |\delta \ln r_\mu| \approx 5.55 \times 10^{-5}, \qquad |\delta \ln r_\tau| \approx 5.07 \times 10^{-5}.$$

A common rescaling of all r_i leaves Q unchanged, so these are shape changes, not total-scale changes. The central FFGFT residual from the quoted inputs,

$$Q_{\text{FFGFT}} - \frac{2}{3} \approx 1.075 \times 10^{-3},$$

is much larger than σ_Q; removing it by changing one coefficient alone would require changes of order 12.3% in r_e, 1.17% in r_μ, or 1.07% in r_τ.

17 Separating D from D_f

Johann Pascher clarified that, in FFGFT, the charged-lepton exponent structure uses the topological torus dimension $D = 3$, not the fractal dimension $D_f = 3 - \xi$. The D_f correction belongs to wavefunction normalisation and related first-order perturbative corrections, not to the charged-lepton exponent ladder itself.

It is important not to mix two different quantities.

- D denotes the topological or spatial dimension used in the torus/counting rule. If $D = 3$ exactly, then
$$\Delta p = \frac{1}{D} = \frac{1}{3}.$$
- D_f denotes the claimed fractal dimension,
$$D_f = 3 - \xi.$$
If the exponent step is tied to D_f instead, then
$$\Delta p_f = \frac{1}{D_f} = \frac{1}{3 - \xi}.$$

The fractal-step expansion is

$$\begin{aligned}\frac{1}{3-\xi} &= \frac{1}{3}\left(1 - \frac{\xi}{3}\right)^{-1} \\ &= \frac{1}{3} + \frac{\xi}{9} + \frac{\xi^2}{27} + O(\xi^3).\end{aligned}$$

The corresponding complement is

$$1 - \frac{1}{3-\xi} = \frac{2}{3} - \frac{\xi}{9} - \frac{\xi^2}{27} + O(\xi^3).$$

With $\xi = 4/30000$, the leading correction is

$$\frac{\xi}{9} = \frac{4}{270000} \approx 1.48148 \times 10^{-5}.$$

Therefore a D_f-based complement route predicts a value slightly below $2/3$:

$$1 - \frac{1}{3-\xi} \approx 0.6666518513.$$

This is distinct from the exact $D = 3$ complement value $2/3$.

Therefore the D_f-based exponent step should be read here as a diagnostic alternative, not as the FFGFT claim. For FFGFT itself, the tau exponent belongs to the exact topological route

$$p_\tau = 1 - \frac{1}{D} = 1 - \frac{1}{3} = \frac{2}{3}.$$

18 FFGFT Exponent Ladder Bookkeeping

The charged-lepton exponent set is

$$p_e = \frac{3}{2}, \qquad p_\mu = 1, \qquad p_\tau = \frac{2}{3}.$$

This is not a uniform ladder. Johann Pascher clarified that this is intentional. The three exponents arise from different torus-geometric mechanisms around the base integer mode:

$$p_\mu = 1,$$

$$p_\tau = 1 - \frac{1}{D} = 1 - \frac{1}{3} = \frac{2}{3},$$

and

$$p_e = 1 + \frac{1}{D-1} = 1 + \frac{1}{2} = \frac{3}{2}.$$

Thus the gaps are

$$p_e - p_\mu = \frac{1}{2}, \qquad p_\mu - p_\tau = \frac{1}{3}.$$

The electron side uses a $1/(D-1)$ spin-winding contribution, while the tau side uses the complement-of-step route $1 - 1/D$. The asymmetry is therefore not a failed uniform 1/3 ladder. It is part of the proposed FFGFT exponent architecture.

In the terminology of this note:

$$p_\tau$$

belongs to Route 2, the complement-of-step route, while

$$p_e$$

belongs to a separate spin-winding route. Neither exponent is to be deformed using D_f in the FFGFT reading.

19 Predictions Under Small Deformation

The routes make different first-order predictions under deformation. This is the clean falsifiability criterion.

Route	Deformed expression	First derivative at the base point
Equal-projector Koide	$Q_{\mathrm{proj}}(3+\epsilon) = 2/(3+\epsilon)$	$-2/9$
Complement step	$p_{\mathrm{comp}}(3+\epsilon) = 1 - 1/(3+\epsilon)$	$+1/9$
Midpoint route	$Q_{\mathrm{mid}}(3+\epsilon) = \frac{1}{2}(1+1/(3+\epsilon))$	$-1/18$
Fractal complement	$p_f(\xi) = 1 - 1/(3-\xi)$	$dp_f/d\xi\vert_0 = -1/9$
FFGFT hybrid rung	$p_\tau(D) = 1 - 1/D$, other quoted exponents fixed	$dQ/d\epsilon \approx -0.0994008$

A claimed mechanism should specify which derivative it predicts. Merely reproducing 2/3 at the undeformed point is not enough to identify the mechanism.

20 Relation Between Koide, FFGFT, and Rank-Split Mediation

The three routes discussed here use different primitive objects:

Route	Primitive object	2/3 mechanism
Koide projector form	Square-root mass vector x	Equal democratic/residual quadratic weights
FFGFT/T0	ξ exponent ladder	Complement of a $1/D$ exponent step
Rank-split mediation	Involutive mediation space	Rank ratio 16/24
Recursive closure	Contractive recursive whole	Complement $1-r$ of unresolved remainder r

They may be related, but the relation has to be demonstrated by a map. The clean FFGFT-to-Koide map is

$$(r_i, p_i, \xi) \longmapsto x_i = \sqrt{r_i}\,\xi^{p_i/2} \longmapsto (A, B) \longmapsto Q.$$

The key test is whether the FFGFT exponent architecture forces

$$A = B,$$

rather than merely fitting it numerically.

21 Conclusion

At $D = 3$, different algebraic roads meet at 2/3:

$$\frac{2}{D}, \qquad 1 - \frac{1}{D}, \qquad \frac{1}{2}\left(1 + \frac{1}{D}\right).$$

But under $D = 3 + \epsilon$ they separate:

$$\begin{aligned}
\frac{2}{3+\epsilon} &= \frac{2}{3} - \frac{2}{9}\epsilon + \frac{2}{27}\epsilon^2 + O(\epsilon^3),\\
1 - \frac{1}{3+\epsilon} &= \frac{2}{3} + \frac{1}{9}\epsilon - \frac{1}{27}\epsilon^2 + O(\epsilon^3),\\
\frac{1}{2}\left(1 + \frac{1}{3+\epsilon}\right) &= \frac{2}{3} - \frac{1}{18}\epsilon + \frac{1}{54}\epsilon^2 + O(\epsilon^3).
\end{aligned}$$

For FFGFT specifically, the sharpened classification is:

$$p_\tau = \frac{2}{3}$$

comes from the Route 2 complement law $1 - 1/D$ at exact topological dimension $D = 3$, while

$$p_e = \frac{3}{2}$$

comes from the distinct spin-winding form $1 + 1/(D-1)$. The value

$$Q_{\text{FFGFT}} \approx 0.6677418577$$

is therefore not an attempted exact derivation of Koide. It is the bare Koide-proximity value of the exact rational FFGFT mass generator.

The deformation is therefore not cosmetic. It is a discriminator. It tells us whether a claimed 2/3 arises from equal projector weight, complement-of-step algebra, midpoint structure, rank splitting, or recursive closure.

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