Recursive Survival Filtering of the Three-Body Problem:

Stability Islands as Closure-Preserving Histories

Peter M. Austin
Information Physics Institute
peter.austin@informationphysicsinstitute.net

June 2025, Revised May 2026

Abstract

The Newtonian three-body problem has no general closed-form solution in elementary functions, and its phase space contains both strongly chaotic regions and regular islands. This note does not claim to solve the general three-body problem in the classical closed-form sense. It proposes a recursive survival-filtering formulation in which candidate histories are generated, transported by a structure-preserving map, projected or polished back toward a constraint surface, and then ranked by invariant residuals and closure defects.

The central object is a recursive state sequence $K_{n+1} = R(K_n)$, specialised to a three-body phase state. The practical update is written

$$K_{n+1} = \Pi_{\mathcal{C}} \Phi_h(K_n),$$

where Φ_h is a numerical Hamiltonian step and $\Pi_{\mathcal{C}}$ is a projection or polishing operation onto the constraint surface carrying the desired invariants. Energy, angular momentum, centre-of-mass, total momentum, and recurrence closure are combined into a residual R_n. A bounded exposure $W_n = R_n/(1 + R_n)$ then defines a recursive survival score

$$S_N = \prod_{n=0}^{N-1} \frac{1}{1 + \Gamma_n W_n h}.$$

Stability islands are interpreted as regions of initial-condition space where invariant drift and closure defect remain small, so histories preserve high survival under the recursive filter.

The construction is related to KAM-style a posteriori validation and polishing of approximately invariant structures, but it does not require action-angle variables as primitive coordinates. It also avoids broad numerical or polytopal overclaims: each claim is expressed in terms of an object, a map, an invariant, and a continuation or test.

1 Introduction

The three-body problem is often introduced as a canonical example of deterministic chaos. Three masses interact through Newtonian gravity, and small changes in their initial conditions can lead to radically different outcomes. This is true, but it is incomplete. The same phase space also contains stable periodic solutions, near-periodic structures, resonant families, and regular islands. The problem is not pure chaos. It is a mixture of chaotic seas, recurrent structures, and partially persistent islands.

This paper develops a disciplined version of a recursive idea: instead of trying to write a single closed-form formula for every possible three-body trajectory, one may generate many histories and ask which histories preserve structure. The result is a filtering scheme. Histories that drift away from the invariants, fail recurrence tests, or require large correction to remain coherent lose survival weight. Histories that preserve invariants and close cleanly retain high survival.

The guiding equation is

$$K_{n+1} = \Pi_{\mathcal{C}} \Phi_h(K_n). \tag{1}$$

Here K_n is the three-body phase state at recursive step n, Φ_h is a finite step of a Hamiltonian integrator, and $\Pi_{\mathcal{C}}$ is a projection or polishing map onto a constraint surface $\mathcal{C}$. The constraint surface carries quantities that should remain stable: energy, centre-of-mass, total momentum, angular momentum, and, where relevant, a periodic or permutation closure class.

The phrase *survival filtering* is used because the method does not merely classify a path as valid or invalid. It assigns a persistence score. A trajectory may be numerically imperfect yet structurally close to a stable island. Conversely, a trajectory may integrate without immediate collision while rapidly losing invariant coherence. The survival score records this difference.

1.1 Claims and non-claims

The paper makes the following limited claims.

(i) The three-body problem can be formulated as a recursive state-generation problem with invariant residuals.

(ii) A structure-preserving update followed by constraint polishing gives a useful diagnostic of coherent histories.

(iii) Stability islands can be represented as high-survival regions in initial-condition space.

(iv) The construction is compatible with KAM-style ideas in which approximately invariant structures can be validated or improved by geometric correction.

The paper does not claim the following.

(i) It does not claim a closed-form solution of the general three-body problem.

(ii) It does not claim that every chaotic trajectory can be made regular by projection.

(iii) It does not claim that a lattice, polytope, or exceptional symmetry group is required for the method.

(iv) It does not claim that the survival score is a new fundamental law of celestial mechanics. It is a diagnostic and modelling construction.

1.2 Methodological rule: object, map, invariant, continuation

A recurring failure mode in speculative mathematical physics is number-chaining: a calculation produces a number, another framework contains a similar number, and the similarity is treated as evidence before the mechanism is fixed. This paper uses a stricter rule.

Every claim is expressed through four components:

Object	The state space or finite structure being studied.
Map	The update, projection, involution, or measurement rule.
Invariant	The quantity preserved, exposed, or tested.
Continuation	The deformation, limit, benchmark, or recurrence test.

For the present paper the object is reduced three-body phase space, the map is $\Pi_{\mathcal{C}}\Phi_h$, the invariants are Hamiltonian and geometric constraints, and the continuation is variation of initial data, step size, period, and symmetry closure class.

Plain-speak. This paper is not trying to win by finding impressive numbers. It asks a simpler question: if we generate a three-body history, how much structure does it keep? The better it keeps energy, angular momentum, centre-of-mass, and closure, the more it survives the filter.

2 Notation Guide

The following notation is used throughout the document. The guide is included early because the framework combines ordinary celestial mechanics with recursive filtering notation.

Symbol	Meaning
i, j	Body indices, usually $i, j \in \{1, 2, 3\}$.
m_i	Mass of body i.
$q_i \in \mathbb{R}^d$	Position of body i, with $d = 2$ or $d = 3$.
$p_i \in \mathbb{R}^d$	Momentum of body i.
$q = (q_1, q_2, q_3)$	Full position vector.
$p = (p_1, p_2, p_3)$	Full momentum vector.
$K = (q, p)$	Full three-body phase state.
K_n	Recursive state at step n.
$\mathcal{K}$	State space of the recursive construction.
$\mathcal{M}_{\text{red}}$	Reduced phase space after imposing centre-of-mass and momentum constraints.
R	Abstract recursive update operator, $K_{n+1} = R(K_n)$.
Φ_h	Numerical Hamiltonian flow step of size h.
$\Pi_{\mathcal{C}}$	Projection or polishing map onto the constraint surface $\mathcal{C}$.
$\mathcal{C}$	Constraint surface carrying chosen invariants.
H	Newtonian Hamiltonian.
H_0	Target initial Hamiltonian.
P_{tot}	Total momentum $\sum_i p_i$.
Q_{cm}	Centre-of-mass position $M^{-1} \sum_i m_i q_i$, where $M = \sum_i m_i$.
L	Total angular momentum.
$\mathcal{G}$	Allowed closure symmetry group, for example identity or body permutations.
C_N	Closure defect after N steps.
R_n	Invariant residual at step n.
$\Delta_{\mathcal{C},n}$	Projection correction needed at step n.
W_n	Bounded residual exposure, $W_n = R_n/(1 + R_n)$.
Γ_n	Non-negative local loss coefficient.
S_N	Survival score after N recursive steps.
$\mathcal{P}_i$	Normalised survival probability assigned to history i.
Θ, Π	Reduced phase coordinates used in the survival-geometry interpretation.
J	Action norm, typically $J = \Theta^2 + \ell^2\Pi^2$.
ℓ	Scale factor placing Θ and Π into comparable units.
ω	Rotation vector or recurrence frequency in KAM-style notation.
$K(\theta)$	Parametrisation of an invariant torus in KAM notation.

3 Classical Three-Body Phase Space

Consider three point masses $m_1, m_2, m_3 > 0$ in d-dimensional Euclidean space. The Newtonian Hamiltonian is

$$H(q,p) = \sum_{i=1}^{3} \frac{\|p_i\|^2}{2m_i} - G \sum_{1\le i<j\le 3} \frac{m_i m_j}{\|q_i - q_j\|}. \tag{2}$$

The canonical equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{3}$$

Equivalently,

$$\ddot{q}_i = G \sum_{j\neq i} m_j \frac{q_j - q_i}{\|q_j - q_i\|^3}. \tag{4}$$

The conserved quantities include the Hamiltonian H, the total momentum

$$P_{\mathrm{tot}} = \sum_{i=1}^{3} p_i, \tag{5}$$

the centre-of-mass position

$$Q_{\mathrm{cm}} = \frac{1}{M} \sum_{i=1}^{3} m_i q_i, \qquad M = \sum_{i=1}^{3} m_i, \tag{6}$$

and the total angular momentum

$$L = \sum_{i=1}^{3} q_i \times p_i. \tag{7}$$

In the planar problem L is represented by a scalar perpendicular to the plane.

A common reduction sets

$$Q_{\mathrm{cm}} = 0, \qquad P_{\mathrm{tot}} = 0, \tag{8}$$

leaving the relative motion. This removes the uninteresting translation of the whole system. Additional reductions by rotation and scaling are possible, but they are not required for the survival-filtering construction. The only requirement is that the chosen state space and constraints be stated explicitly.

3.1 Closure and recurrence

A history is periodic if, after some time $T = Nh$, the state returns to its initial value:

$$K_N = K_0. \tag{9}$$

For choreographies and symmetric periodic orbits, exact return may occur only after a body permutation or spatial symmetry. We therefore define closure relative to a finite symmetry set $\mathcal{G}$:

$$C_N(K_0) = \min_{\sigma\in\mathcal{G}} \frac{\|K_N - \sigma K_0\|_\Lambda}{1 + \|K_0\|_\Lambda}. \tag{10}$$

The weighted norm $\|\cdot\|_\Lambda$ is chosen to compare position and momentum components in compatible units. The identity element in $\mathcal{G}$ gives ordinary periodic closure. Nontrivial permutations allow cyclic body exchange, as in choreographic solutions.

Plain-speak. The normal equations describe where the three masses go. The extra closure test asks whether the whole pattern returns to itself, perhaps after relabelling the bodies. That is important because some beautiful three-body solutions are not just repeated positions, but repeated patterns with the bodies taking each other's places.

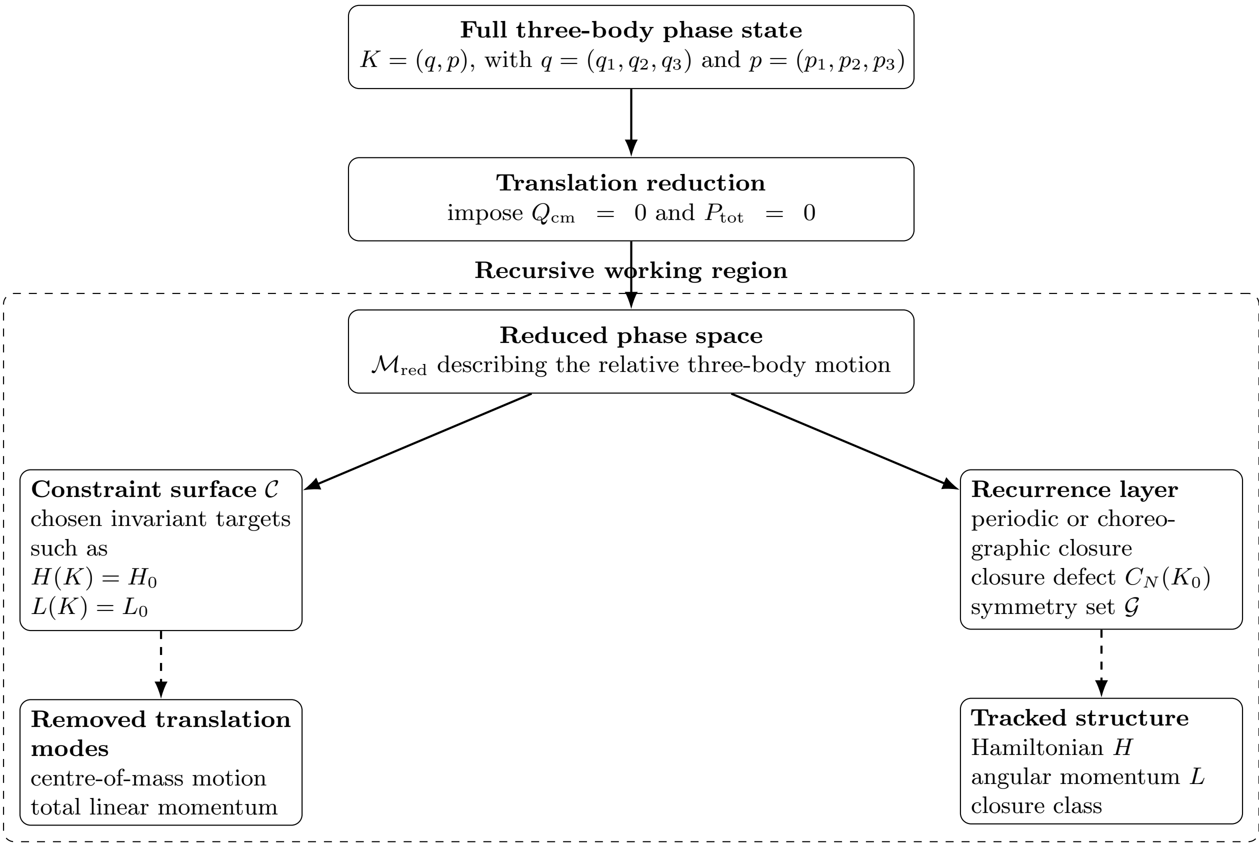


Figure 1: Reduced three-body phase space showing the full state (q, p), reduction by centre-of-mass and total-momentum constraints, the reduced phase space $\mathcal{M}_{\rm red}$, the constraint surface $\mathcal{C}$, and the recurrence layer used for periodic or choreographic closure tests.

4 Recursive State Generation

The abstract recursive form is

$$K_{n+1} = R(K_n), \tag{11}$$

where $K_n \in \mathcal{K}$ is a state at recursive depth n. For three-body dynamics, the state is the phase point

$$K_n = (q_{1,n}, q_{2,n}, q_{3,n}, p_{1,n}, p_{2,n}, p_{3,n}). \tag{12}$$

The recursive view is not a different force law. It is a different way of organising the computation. A history is the ordered sequence

$$\gamma(K_0) = \{K_0, K_1, K_2, \ldots\}. \tag{13}$$

A family of histories is generated by sampling a family of initial states $K_0^{(a)}$ and evolving each one under the same rule.

4.1 Structure-preserving update

The practical update is decomposed into two operations:

$$K_{n+1} = \Pi_{\mathcal{C}} \Phi_h(K_n). \tag{14}$$

Here Φ_h is a finite-time step of a Hamiltonian numerical method, preferably a symplectic or near-symplectic scheme. The map $\Pi_{\mathcal{C}}$ is a projection, correction, or polishing operation onto the chosen constraint surface $\mathcal{C}$.

The constraint surface may be written schematically as

$$\mathcal{C} = \{K : H(K) = H_0,\ P_{\text{tot}}(K) = P_0,\ Q_{\text{cm}}(K) = Q_0,\ L(K) = L_0\}. \tag{15}$$

For periodic searches, $\mathcal{C}$ may be extended by a closure condition, for example $C_N \approx 0$.

Projection is not free. It may change the state, and that change must be measured. Define the correction vector

$$\Delta_{\mathcal{C},n} = \Pi_{\mathcal{C}}\Phi_h(K_n) - \Phi_h(K_n). \tag{16}$$

A small correction means the raw Hamiltonian step already lies near the desired structure. A large correction means the history is being forced into coherence and should lose survival weight.

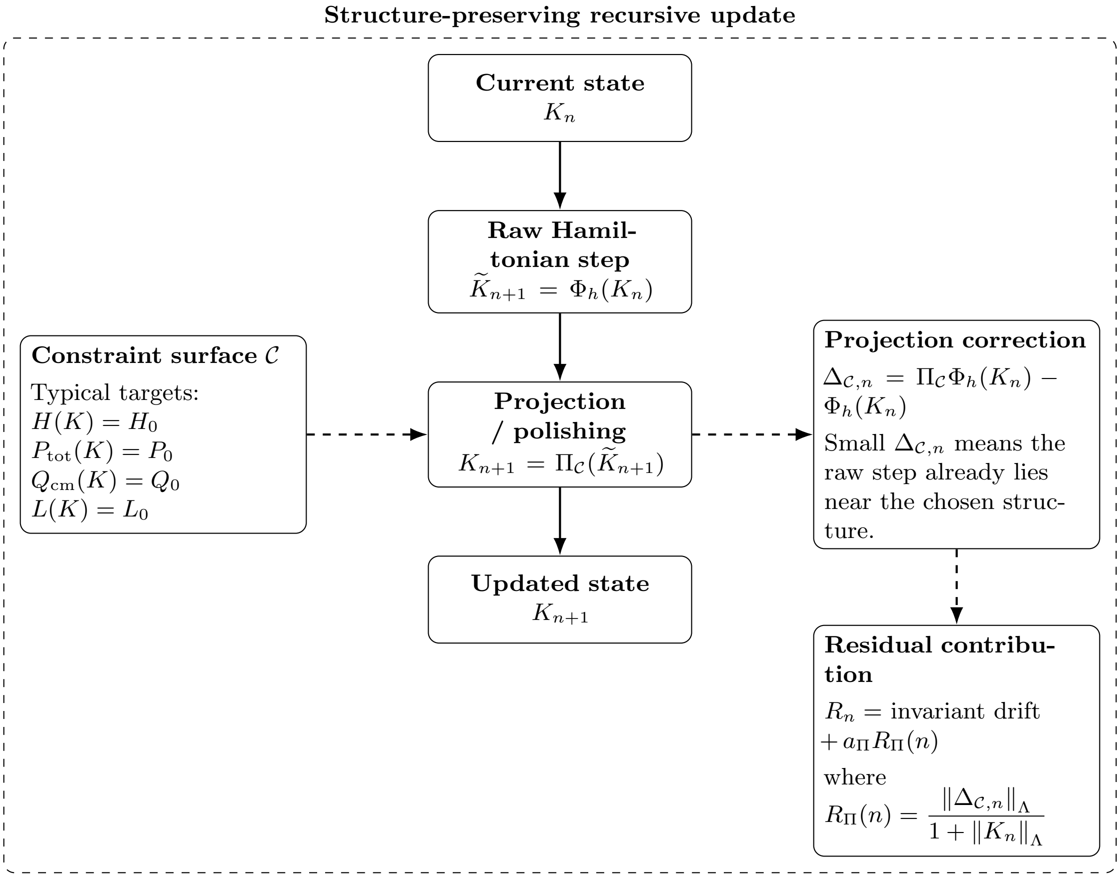


Figure 2: Structure-preserving recursive update. A current state K_n is advanced by a Hamiltonian step Φ_h to produce the raw state $\widetilde{K}_{n+1}$, then projected or polished by $\Pi_{\mathcal{C}}$ onto the constraint surface $\mathcal{C}$ to obtain K_{n+1}. The correction vector $\Delta_{\mathcal{C},n}$ measures how much the raw step had to be adjusted, and contributes to the residual used in survival filtering.

4.2 Recursive closure as a fixed-point principle

A closed recursive object is not merely a long list of completed steps. It is a state sequence satisfying a recurrence rule and a closure condition. The finite periodic case is

$$K_N = \sigma K_0 \tag{17}$$

for some $\sigma \in \mathcal{G}$. In fixed-point form, this is

$$F_N(K_0) = \sigma K_0, \qquad F_N = (\Pi_{\mathcal{C}}\Phi_h)^N. \tag{18}$$

Thus periodic orbits are fixed points of a return map, possibly after a symmetry action.

This connects to the recursive scalar closure law used elsewhere in the framework. A scalar recursion of the form

$$S = a + rS \tag{19}$$

closes when $|r| < 1$, giving

$$S = \frac{a}{1-r}. \tag{20}$$

The three-body case is not a scalar geometric series, but the principle is analogous: an apparently extended process becomes closed when the generating rule and the return condition stabilise together.

Plain-speak. We generate the path step by step, but the real object is the closed rule. A periodic orbit is not just many positions drawn in a row. It is a loop in state space that survives the return test.

5 Invariant Residuals and Survival Weighting

The residual is the central diagnostic. It measures how much of the intended structure is lost at each recursive step.

5.1 Invariant residual components

Let K_n be the state after projection at step n. Define the energy residual

$$R_E(n) = \frac{|H(K_n) - H_0|}{1 + |H_0|}. \tag{21}$$

Define the total momentum residual

$$R_P(n) = \frac{\|P_{\text{tot}}(K_n) - P_0\|}{1 + \|P_0\|}, \tag{22}$$

the centre-of-mass residual

$$R_Q(n) = \frac{\|Q_{\text{cm}}(K_n) - Q_0\|}{1 + \|Q_0\|}, \tag{23}$$

and the angular momentum residual

$$R_L(n) = \frac{\|L(K_n) - L_0\|}{1 + \|L_0\|}. \tag{24}$$

The projection correction residual is

$$R_\Pi(n) = \frac{\|\Delta_{\mathcal{C},n}\|_\Lambda}{1 + \|K_n\|_\Lambda}. \tag{25}$$

A combined residual is then

$$R_n = a_E R_E(n) + a_P R_P(n) + a_Q R_Q(n) + a_L R_L(n) + a_\Pi R_\Pi(n), \tag{26}$$

where all coefficients $a_E, a_P, a_Q, a_L, a_\Pi \geq 0$ are chosen before the computation.

For a finite return search, append the closure residual

$$R_N^{\text{total}} = \frac{1}{N}\sum_{n=0}^{N-1} R_n + a_C C_N. \tag{27}$$

The closure coefficient a_C should also be fixed before testing.

5.2 Bounded exposure

The residual R_n may be unbounded. To turn it into a bounded exposure factor, define

$$W_n = \frac{R_n}{1 + R_n}. \tag{28}$$

Then

$$0 \leq W_n < 1. \tag{29}$$

Low residual gives $W_n \approx 0$. Large residual gives $W_n \approx 1$. This makes W_n a local exposure to structural loss.

This is a three-body version of the more general survival-geometry split between generation and selection. In the phase-space model, one defines an action norm

$$J = \Theta^2 + \ell^2 \Pi^2 \tag{30}$$

and an exposure weight

$$W^{\Theta} = \frac{\Theta^2}{J}. \tag{31}$$

For the three-body diagnostic, the exposure is not positional dominance but invariant residual exposure. The shared idea is that histories are generated first and filtered second.

5.3 Recursive survival product

Define the survival score recursively by

$$S_{n+1} = \frac{S_n}{1 + \Gamma_n W_n h}, \qquad S_0 = 1, \tag{32}$$

where $\Gamma_n \geq 0$ is a local loss coefficient and $h > 0$ is the step size. After N steps,

$$S_N = \prod_{n=0}^{N-1} \frac{1}{1 + \Gamma_n W_n h}. \tag{33}$$

This form uses only recursive multiplication and division. No logarithmic or exponential expression is needed as primitive notation.

For a family of generated histories γ_i, the normalised representation weight is

$$\mathcal{P}_i = \frac{S_i}{\sum_j S_j}. \tag{34}$$

Thus the most represented histories are the ones that lose the least survival weight under the residual filter.

Plain-speak. The survival score is a way of saying how hard the trajectory has to be corrected. If a path preserves the important quantities naturally, it keeps a high score. If it needs constant repair, it fades from the ranked set.

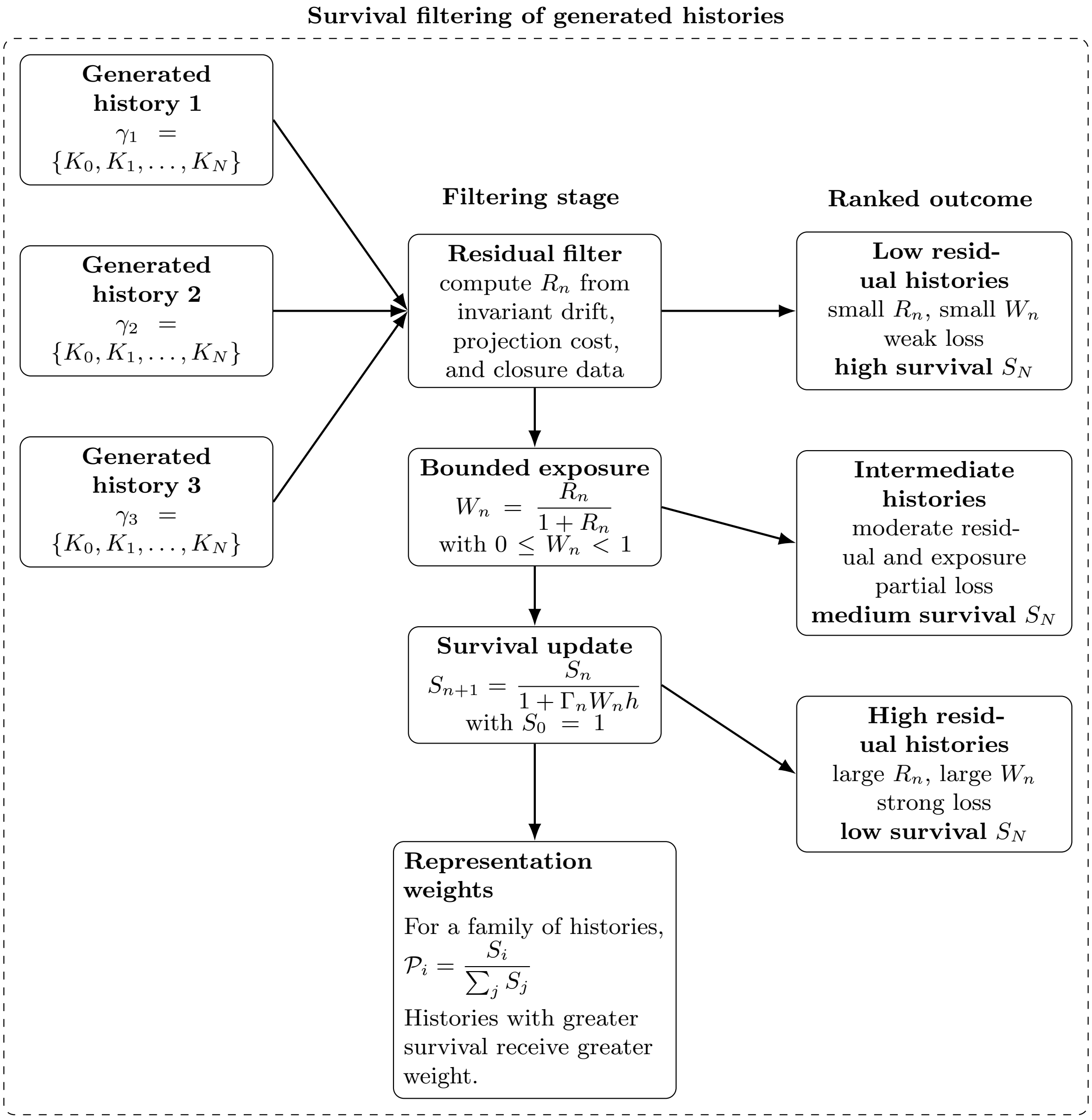


Figure 3: Survival filtering of generated histories. Candidate histories are evaluated by a residual filter, converted into bounded exposure values $W_n = R_n/(1 + R_n)$, and recursively updated by the survival rule $S_{n+1} = S_n/(1 + \Gamma_n W_n h)$. Histories that preserve structure retain high survival, while histories with large residuals lose weight.

6 Stability Islands as Closure-Preserving Histories

A stability island is a region of initial-condition space in which nearby histories remain coherent for long times. In this paper coherence means four things:

(i) invariant residuals remain small,

(ii) closure defects remain small for candidate periods or recurrence windows,

(iii) projection corrections remain small,

(iv) survival scores remain high across nearby initial conditions.

Let $U \subset \mathcal{M}_{\mathrm{red}}$ be a set of initial conditions. Define the survival field

$$\mathscr{S}_N : U \to [0,1], \qquad \mathscr{S}_N(K_0) = S_N(K_0). \tag{35}$$

Define also the closure field

$$\mathscr{C}_N : U \to [0,\infty), \qquad \mathscr{C}_N(K_0) = C_N(K_0). \tag{36}$$

A practical stability island at horizon N may then be defined as

$$\mathcal{I}_N(s_*, c_*) = \{K_0 \in U : \mathscr{S}_N(K_0) \geq s_*,\ \mathscr{C}_N(K_0) \leq c_*\}, \tag{37}$$

where s_* and c_* are thresholds chosen before scanning.

This definition does not require an exact analytic torus, but it can detect one if present. It also allows finite-time islands, quasi-periodic structures, and stable recurrent patterns to be ranked without asserting that all surrounding chaotic behaviour has been solved.

6.1 Local persistence and neighbourhood stability

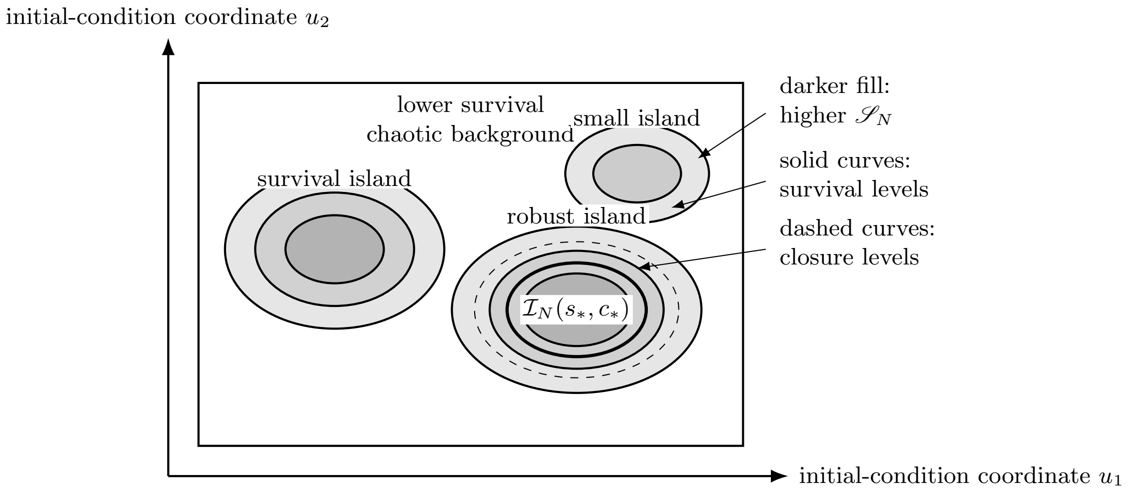
A single high-survival history can be a numerical accident. A stability island should persist under small perturbations. Therefore one should also compute a neighbourhood statistic, for example

$$\overline{S}_{N,\rho}(K_0) = \frac{1}{\mathrm{vol}(B_\rho(K_0))} \int_{B_\rho(K_0)} S_N(K)\, \mathrm{d}K, \tag{38}$$

where $B_\rho(K_0)$ is a small ball in initial-condition space. A robust island has high S_N and high $\overline{S}_{N,\rho}$. A brittle isolated path may have high S_N but low neighbourhood survival.

Schematic survival-island map on an initial-condition plane. Darker regions indicate higher survival $\mathscr{S}_N$, while dashed curves represent closure structure. The highlighted set $\mathcal{I}_N(s_*, c_*)$ marks a practical stability island, where survival remains high and closure defect remains small.

Plain-speak. One nice orbit is not enough. We want patches of nearby starts that also behave well. A stability island is a whole neighbourhood where the filter stays gentle.



Interpretation
A practical stability island is the subset of initial conditions for which the survival field stays above a chosen threshold and the closure field stays below a chosen threshold:

$$\mathcal{I}_N(s_*, c_*) = \left\{ K_0 \in U : \begin{array}{c} \mathscr{S}_N(K_0) \geq s_*, \\ \mathscr{C}_N(K_0) \leq c_* \end{array} \right\}.$$

Figure 4: Schematic survival-island map on an initial-condition plane. Darker regions indicate higher survival $\mathscr{S}_N$, while dashed curves represent closure structure. The highlighted set $\mathcal{I}_N(s_*, c_*)$ marks a practical stability island, where survival remains high and closure defect remains small.

7 Connection to KAM-Style Polishing

KAM theory studies the persistence of invariant tori in Hamiltonian systems. A useful modern formulation avoids requiring the Hamiltonian to be explicitly written in action-angle variables. Instead, it begins with an approximately invariant torus and shows, under non-degeneracy and Diophantine conditions, that a true invariant torus lies nearby.

In map notation, an invariant torus is parametrised by

$$K : \mathbb{T}^m \to U \tag{39}$$

and satisfies

$$f \circ K(\theta) = K(\theta + \omega). \tag{40}$$

An approximate torus has an error

$$E(\theta) = f \circ K(\theta) - K(\theta + \omega). \tag{41}$$

The polishing idea is that when E is small enough and the torus is not too degenerate, a correction scheme can improve K toward a true invariant torus.

The survival-filtering construction is not a KAM theorem. It is a finite-history analogue. The correspondence is:

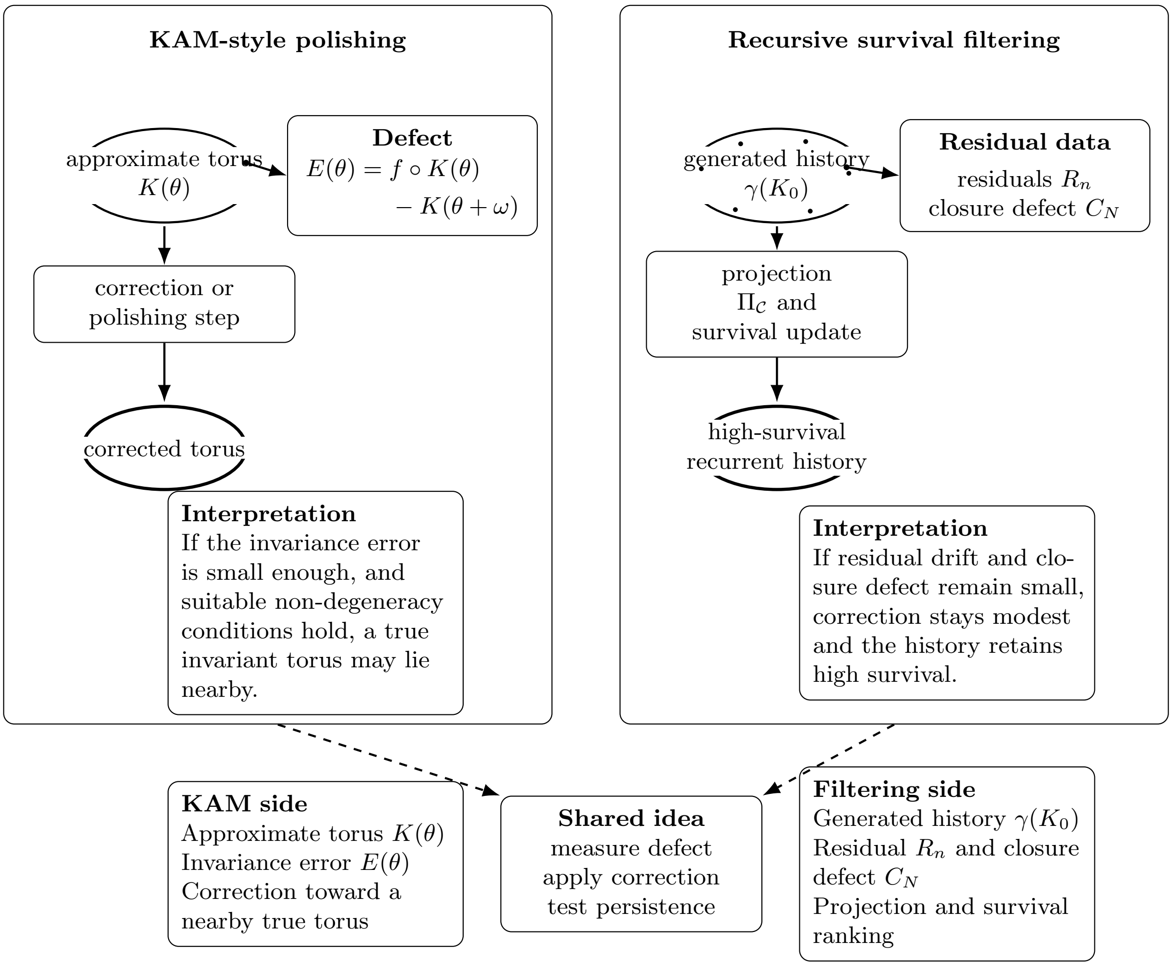


Figure 5: Comparison between KAM-style polishing and recursive survival filtering. On the left, an approximate invariant torus is assessed by its invariance defect and corrected toward a true torus. On the right, a generated history is assessed by residuals and closure defect, then projected and ranked by survival. In both cases the common logic is to measure structural defect, apply disciplined correction, and test persistence.

KAM polishing	Survival filtering
Approximate torus $K(\theta)$	Generated history $\gamma(K_0)$
Invariance error $E(\theta)$	Residual R_n and closure defect C_N
Correction of K	Projection $\Pi_{\mathcal{C}}$ and correction $\Delta_{\mathcal{C},n}$
Non-degeneracy condition	Robust neighbourhood survival
True torus nearby	High-survival island or polished recurrence

The conceptual debt is clear: the purpose is not to transform the system into ideal coordinates first, but to use the geometry of the state space to correct and validate structures where they are found.

Plain-speak. KAM polishing says, roughly, that if you have an almost-invariant shape and it is not too badly behaved, there may be a true invariant shape nearby. Our method borrows that attitude: do not demand perfect coordinates first. Measure the defect, correct it, and see whether the structure persists.

8 Algorithmic Skeleton

The method can be implemented as a numerical diagnostic. This section gives an algorithmic skeleton rather than a specific codebase.

8.1 Inputs

Choose:

(i) masses m_1, m_2, m_3,

(ii) a region $U \subset \mathcal{M}_{\mathrm{red}}$ of initial conditions,

(iii) a Hamiltonian stepper Φ_h,

(iv) a constraint projection $\Pi_{\mathcal{C}}$,

(v) a symmetry closure set $\mathcal{G}$,

(vi) residual coefficients $a_E, a_P, a_Q, a_L, a_\Pi, a_C$,

(vii) loss coefficients Γ_n or a rule for assigning them,

(viii) a horizon N.

8.2 Procedure

For each initial state $K_0 \in U$:

1. Set $S_0 = 1$.
2. For $n = 0, 1, \ldots, N-1$:
   a. Compute the raw step $\widetilde{K}_{n+1} = \Phi_h(K_n)$.
   b. Project or polish $K_{n+1} = \Pi_{\mathcal{C}}(\widetilde{K}_{n+1})$.
   c. Compute $\Delta_{\mathcal{C},n} = K_{n+1} - \widetilde{K}_{n+1}$.
   d. Compute residual components and combine them into R_n.
   e. Set $W_n = R_n/(1 + R_n)$.
   f. Update $S_{n+1} = S_n/(1 + \Gamma_n W_n h)$.
3. Compute final closure defect C_N.
4. Record S_N, C_N, R_N^{total}, and any collision or escape status.

8.3 Pseudocode

```
for K0 in initial_condition_grid:
    K = K0
    S = 1
    for n in range(N):
        K_raw = Phi_h(K)
        K_next = project_to_constraints(K_raw)
        Delta = K_next - K_raw
        R = residual(K_next, Delta, targets)
        W = R / (1 + R)
        S = S / (1 + Gamma(n, K_next) * W * h)
```

```
        K = K_next
    C = closure_defect(K, K0, symmetry_group)
    record(K0, S, C, R)
```

Plain-speak. The program is simple in essence. Start many possible systems, evolve them, repair only according to pre-declared rules, charge the repair as residual, and mark the histories that need the least repair.

9 Benchmarks

A framework of this kind must be tested against known structures before it is used speculatively. The following benchmarks are natural.

9.1 Euler collinear solutions

Euler solutions are collinear relative equilibria. They provide simple benchmark orbits with known symmetry and constrained geometry. A survival filter should identify their recurrence structure when initial conditions are set near the corresponding family. It should also show loss of survival when transverse perturbations push the system away from the collinear structure.

9.2 Lagrange equilateral solutions

Lagrange solutions keep the three bodies in an equilateral triangle while rotating. They are important because they provide a stable geometric pattern in which the shape is preserved. The survival filter should score exact or near-exact Lagrange initial conditions highly under suitable closure tests.

9.3 Chenciner-Montgomery figure-eight orbit

The figure-eight solution is a periodic equal-mass planar orbit in which the three bodies chase one another around the same eight-shaped curve. It has zero angular momentum and a rich symmetry pattern. In the present framework it is a primary test for permutation closure. The correct closure set $\mathcal{G}$ should include cyclic body relabelling, because the pattern closes as a choreography rather than as three independent curves.

For a candidate period N, the figure-eight closure test becomes

$$C_N^{(8)} = \min_{\sigma \in C_3} \frac{\|K_N - \sigma K_0\|_\Lambda}{1 + \|K_0\|_\Lambda}, \tag{42}$$

where C_3 is the cyclic permutation group of the three bodies. A successful implementation should show a low closure defect and high survival score near the known figure-eight initial condition.

9.4 Regular islands in scattering phase space

Modern numerical studies of three-body scattering identify regular islands inside the broader chaotic phase space. The survival-filtering framework is designed for precisely this mixture. It should not classify the whole system as chaotic or regular. It should produce a graded map: chaotic regions have low survival and high residual, while regular islands have high survival and low residual.

A useful test is to compare a survival-island map against an independent chaos indicator such as Lyapunov time, frequency-map analysis, recurrence statistics, or escape-time sensitivity.

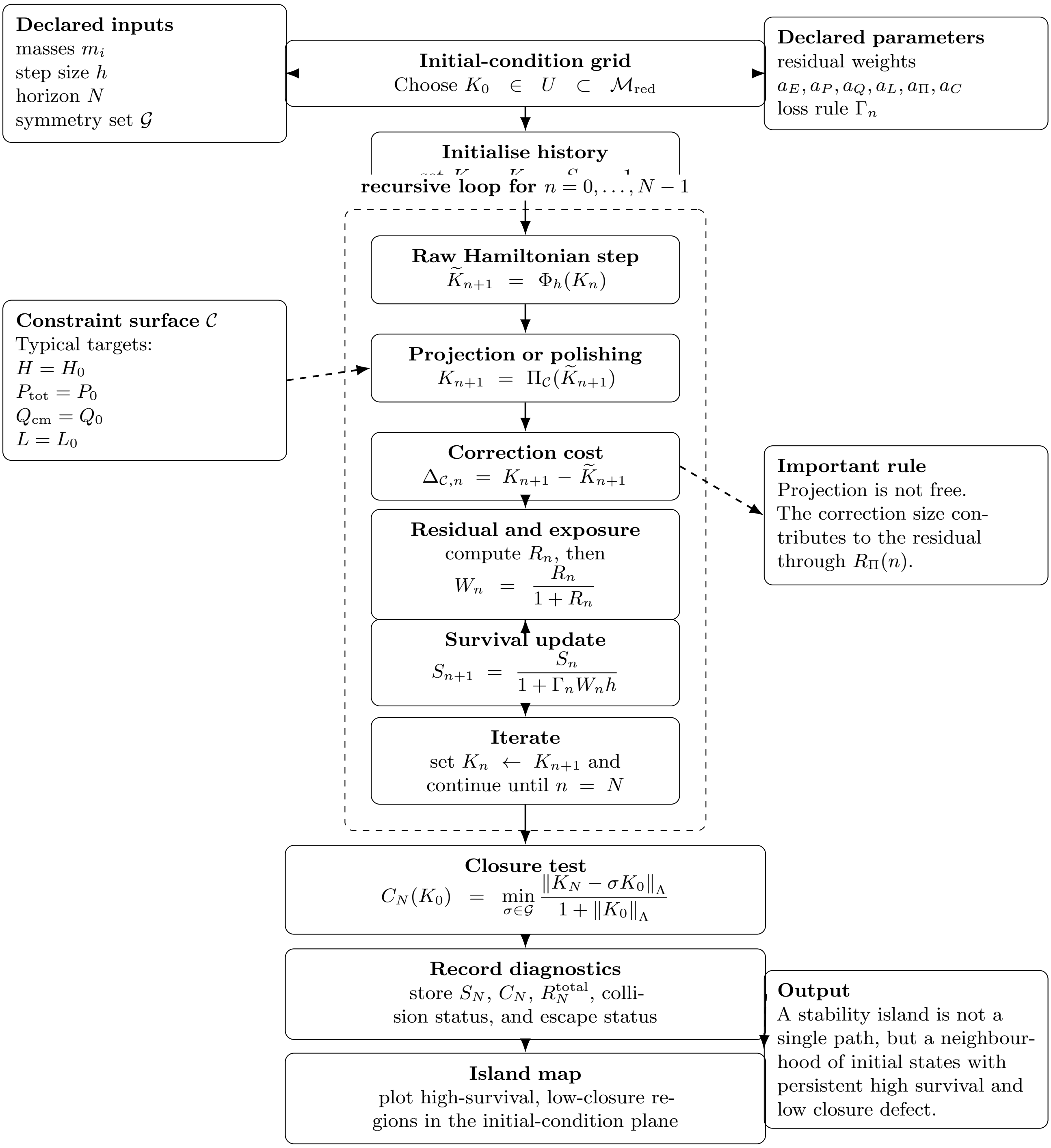


Figure 6: Algorithmic flowchart for recursive survival filtering. Each initial state K_0 is evolved by a Hamiltonian step, projected or polished onto the declared constraint surface, charged for its correction cost, ranked by residual exposure, and updated by the survival rule. After the recursive horizon N, the closure defect and survival score are recorded to build an island map in initial-condition space.

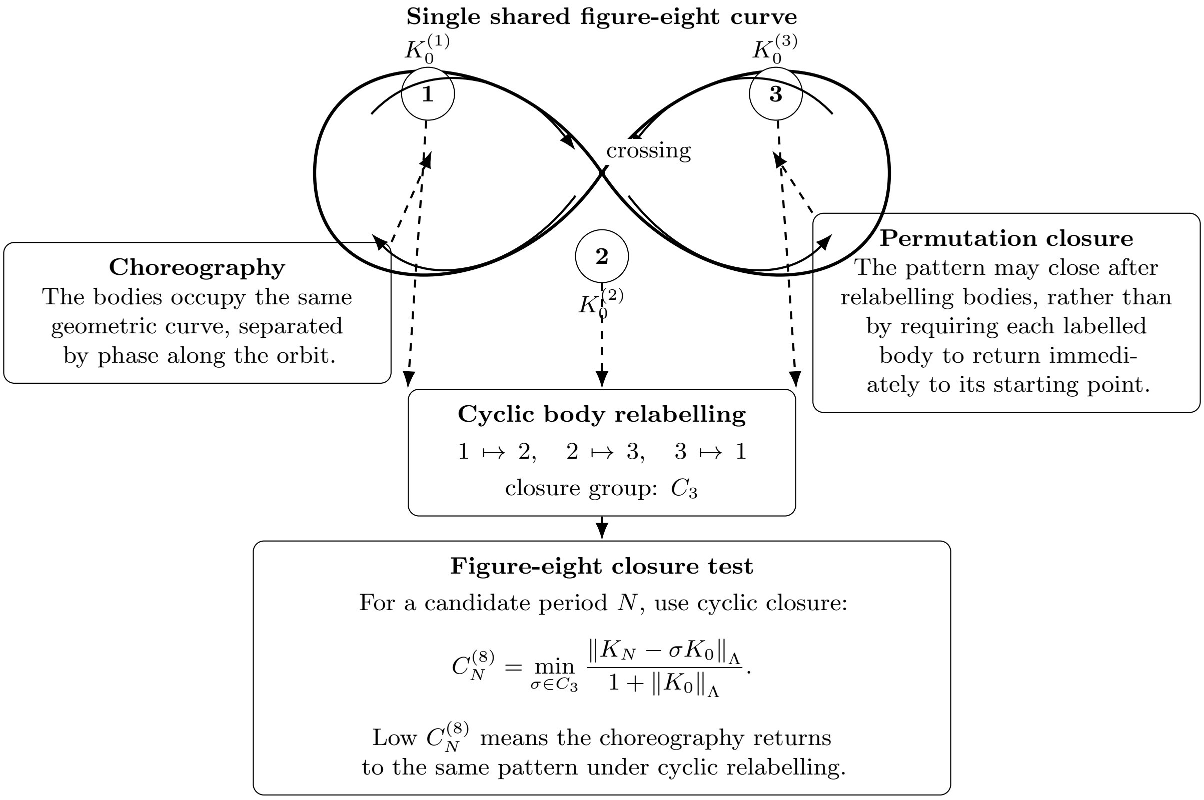


Figure 7: Chenciner-Montgomery figure-eight choreography benchmark. Three equal masses follow one shared figure-eight curve with phase separation. The appropriate recurrence test is cyclic permutation closure, so the pattern is compared against σK_0 for $\sigma \in C_3$ rather than only against the identity return K_0.

Agreement would support the diagnostic. Disagreement would expose where the residual definition is missing a relevant invariant or where the survival filter is too permissive.

Plain-speak. The benchmark list keeps the paper honest. First check the famous simple patterns. Then check known regular islands. Only after that should the method be used for new claims.

10 Relation to No-Sine Phase Geometry

The phase-space part of the survival model often has oscillator-like equations. It is tempting to describe such motion with sine and cosine, but those functions need not be primitive. The circular phase geometry can be generated by perpendicular recursion.

Let

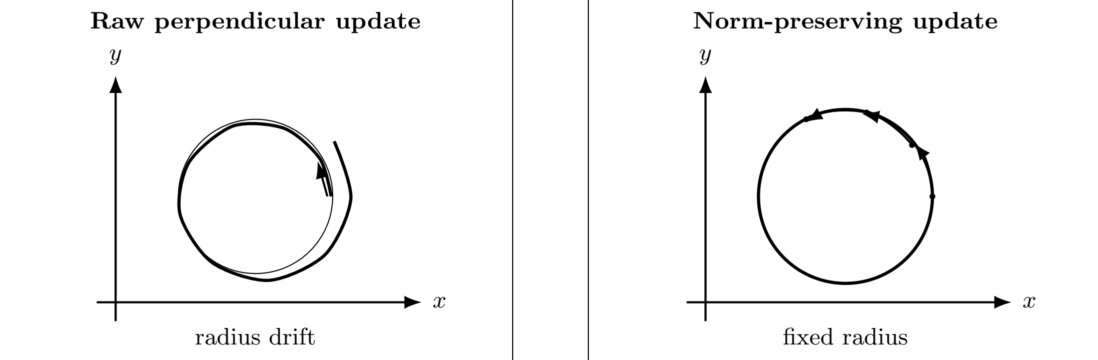
$$z_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \tag{43}$$

and define $q = \omega h$. The raw perpendicular update is

$$\widetilde{x}_{n+1} = x_n - q y_n, \qquad \widetilde{y}_{n+1} = y_n + q x_n. \tag{44}$$

This has the correct perpendicular coupling but produces radius drift:

$$\widetilde{x}_{n+1}^2 + \widetilde{y}_{n+1}^2 = (1 + q^2)(x_n^2 + y_n^2). \tag{45}$$



Raw update

$$\tilde{x}_{n+1} = x_n - q y_n, \qquad \tilde{y}_{n+1} = y_n + q x_n$$

The coupling is perpendicular, but the norm grows:

$$\tilde{r}^2_{n+1} = (1+q^2) r^2_n.$$

Normalised update

Let $z_n = (x_n, y_n)^T$. Then

$$z_{n+1} = \frac{1}{\sqrt{1+q^2}} \begin{pmatrix} 1 & -q \\ q & 1 \end{pmatrix} z_n.$$

This gives $r^2_{n+1} = r^2_n$.

Phase interpretation

With $x = \Theta$ and $y = \Pi/\Omega$, the preserved circle corresponds to

$$\Pi^2 + \Omega^2\Theta^2 = \Omega^2 R^2.$$

Thus circular phase geometry is generated by an algebraic recursion, without taking sine or cosine as primitive.

Figure 8: No-sine circular phase recursion. The raw perpendicular update has the correct coupling but produces radial drift. Dividing by $\sqrt{1+q^2}$ gives an algebraic norm-preserving recursion, generating circular phase geometry without taking trigonometric functions as primitive.

A norm-preserving algebraic update is obtained by dividing by $\sqrt{1+q^2}$:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \frac{1}{\sqrt{1+q^2}} \begin{pmatrix} 1 & -q \\ q & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}. \tag{46}$$

Then

$$x^2_{n+1} + y^2_{n+1} = x^2_n + y^2_n. \tag{47}$$

For phase variables, set

$$x = \Theta, \qquad y = \frac{\Pi}{\Omega} \tag{48}$$

when $\Omega > 0$ is locally constant. The preserved circle

$$x^2 + y^2 = R^2 \tag{49}$$

is equivalent to

$$\Pi^2 + \Omega^2\Theta^2 = \Omega^2 R^2. \tag{50}$$

This is the quadratic form used in the lossless energy diagnostic.

Plain-speak. You do not need to begin with sine and cosine. Circular motion can be built from the rule: x changes by minus y, y changes by x, and the radius is normalised. That is useful because the survival framework is recursive from the start.

11 What Survives from the Older K-Line Version

The older K-Line three-body note contained several useful ideas: discrete paths, recursive updates, norm preservation by orthogonalisation, and the interpretation of islands of regularity as stability nodes. It also used language that is too strong for a rigorous presentation, including claims of resolving the whole three-body problem. The present note keeps the useful core and removes the overclaim.

The retained elements are:

(i) trajectories are represented as recursive histories,

(ii) stability is linked to norm and invariant preservation,

(iii) regular islands are treated as structured recurrence rather than noise,

(iv) recursive phase geometry is built algebraically rather than by imposing trigonometric parametrisation.

The discarded or softened elements are:

(i) no claim of a general closed-form solution,

(ii) no identification of every stable region with a specific high-dimensional polytope,

(iii) no dependence on E8, H4, or 24-cell structure,

(iv) no claim that gravity and magnetism have been unified by the three-body diagnostic.

The revised statement is therefore:

> The three-body problem admits recursive survival filtering: a way to generate candidate histories, measure how well they preserve Hamiltonian structure, and rank their recurrence as stability islands.

Plain-speak. The old version had the right instinct but tried to carry too much. The upgraded version keeps the engine: recursion, invariant preservation, closure, and survival. It drops the grander claims until they earn their place.

12 Limitations and Failure Modes

A survival-filtering method can fail in several ways:

12.1 Projection can hide numerical error

If $\Pi_{\mathcal{C}}$ is too aggressive, it may force a trajectory to look coherent even when the raw dynamics are not. This is why R_Π must be charged as part of the residual. A high-survival trajectory should require small correction, not large correction.

The exact construction of $\Pi_{\mathcal{C}}$ is implementation-dependent and must be reported with each numerical experiment. Different projection rules may preserve the same nominal constraints while introducing different correction geometry, so $\Pi_{\mathcal{C}}$ is part of the tested method, not an invisible preprocessing step.

12.2 A finite-time island may not persist

A trajectory may look regular over a short horizon but become chaotic later. Therefore one should scan survival over increasing N, and report the horizon. A claim of stability must specify the tested time scale.

12.3 Thresholds can create artificial islands

The thresholds s_* and c_* in (37) must be declared before scanning. If they are adjusted after looking at the data, the resulting island map may be biased.

12.4 Collision and near-collision regularisation

Three-body dynamics contain singularities at collision. Regularisation methods may be required near collision configurations. This paper does not provide a new collision regularisation theorem. If regularisation is used, it must be included in the object-map-invariant-continuation description.

12.5 Survival is a diagnostic, not a probability law by itself

The normalised survival weight $\mathcal{P}_i$ is a representation weight over the generated history set. It is not automatically a physical probability unless the generation measure and sampling procedure are specified.

Plain-speak. The filter can fool us if we let it. The correction must be counted. The time horizon must be stated. The thresholds must be fixed in advance. Survival is a ranking method until a physical sampling law is supplied.

13 Possible Tests

The framework becomes useful only if it survives comparison with established numerical and analytic structures. The following tests are proposed.

13.1 Invariant-drift test

Compare Φ_h alone with $\Pi_{\mathcal{C}}\Phi_h$. The projected method should reduce invariant drift, but the total projection correction $\sum_n R_\Pi(n)$ must remain small for a high-survival classification.

13.2 Known-orbit recovery

Starting near known Euler, Lagrange, and figure-eight configurations, the filter should recover high survival and low closure defect. If it fails on these known structures, the residual definition is insufficient.

13.3 Island-map comparison

Construct a grid in initial-condition space and compute $\mathscr{S}_N$. Compare high-survival regions with independent regularity diagnostics. The map should identify islands without swallowing the surrounding chaotic sea.

13.4 Step-size continuation

Repeat the computation for decreasing h. A genuine stability island should not disappear merely because the step size changes. The survival score may change, but the location and shape of robust islands should converge.

Benchmark	**Invariant drift**	**Closure defect**	**Survival and neighbourhood**	**Continuation check**
Euler collinear solutions	Low near the family	Periodic or relative closure	High if aligned; transverse perturbations should reduce survival	Vary perturbation
Lagrange equilateral solutions	Low for exact data	Clean rotational closure	High near the preserved triangular pattern	Vary step size h
Figure-eight choreography	Low near the known orbit	Low under cyclic C_3 closure	High near the choreographic recurrence	Change $\mathcal{G}$
Regular islands in scattering phase space	Low inside islands	Depends on recurrence window N	High over patches, not just isolated paths	Compare chaos indicators
Step-size continuation	Should converge	Should stabilise	Island shape should persist	Take $h \downarrow 0$
Symmetry-class continuation	Mostly unchanged locally	Sensitive to $\mathcal{G}$	Survival changes if closure class changes	Identity versus permutation

Use of the matrix. The filter should first recover known coherent structures: Euler collinear solutions, Lagrange equilateral solutions, and the Chenciner-Montgomery figure-eight. It should then identify regular islands without swallowing the chaotic sea. Step-size continuation and symmetry-class continuation test whether the result is a stable diagnostic rather than a numerical artefact.

Figure 9: Benchmark testing matrix for recursive survival filtering. Each benchmark is assessed against invariant drift, closure behaviour, survival and neighbourhood persistence, and continuation behaviour. The matrix keeps the method tied to known structures before it is used to classify new survival islands.

13.5 Symmetry-class continuation

For choreographies, change the closure group $\mathcal{G}$. A figure-eight search should respond strongly to cyclic permutation closure. An ordinary periodic orbit should be less sensitive to body relabelling. This tests whether closure class, rather than only low residual, is doing real work.

Plain-speak. A good idea should pass boring tests. Known orbits first, then grid maps, then step-size changes, then symmetry checks.

14 Conclusion

This paper has proposed a recursive survival-filtering formulation of the three-body problem. It does not replace the Newtonian equations and does not claim a closed-form solution. Instead, it adds a diagnostic layer above ordinary Hamiltonian evolution.

The essential construction is

$$K_{n+1} = \Pi_{\mathcal{C}} \Phi_h(K_n),$$

with invariant residuals, closure defects, bounded exposure, and recursive survival weighting. Stability islands are then described as high-survival, low-residual, closure-preserving histories in initial-condition space.

The value of the formulation is methodological. It gives a disciplined way to talk about regular islands without collapsing the entire three-body problem into either chaos or exact solvability. It also offers a bridge between recursive survival geometry and KAM-style polishing, while avoiding unsupported numerical or polytopal overclaims.

The next step is computational: implement the algorithm, test it against Euler, Lagrange, and figure-eight benchmarks, and compare the resulting survival maps with known regular-island studies.

A Recursive Closure and Zeno-Type Fixed Points

The scalar closure rule

$$S = a + rS \tag{51}$$

solves to

$$S = \frac{a}{1-r} \tag{52}$$

when $|r| < 1$. For a unit target L, choosing the first resolved segment $(1-r)L$ gives

$$S = (1-r)L + rS, \tag{53}$$

and therefore

$$S = L. \tag{54}$$

This holds for $r = 1/2$, $r = 1/3$, $r = 2/3$, or any contraction $0 < r < 1$. The infinite process closes because the unresolved remainder tends to zero.

For variable ratios r_n, define

$$R_0 = 1, \qquad R_{n+1} = R_n r_n. \tag{55}$$

Let the resolved piece be

$$p_n = L(1-r_n)R_n. \tag{56}$$

Then the partial sum telescopes:

$$\sum_{n=0}^{N} p_n = L(1 - R_{N+1}). \tag{57}$$

If $R_N \to 0$, then

$$\sum_{n=0}^{\infty} p_n = L. \tag{58}$$

This appendix is included because the three-body closure test uses the same conceptual move: closure is not the completion of infinitely many isolated tasks. It is the fixed point of a recursive rule under a contraction or return condition.

B Minimal Python-Like Pseudocode

```
def survival_filter(K0, N, h, targets, params):
    K = K0
    S = 1.0
    history = [K0]

    for n in range(N):
        K_raw = Phi_h(K, h)
```

```
        K_next = Pi_C(K_raw, targets)
        Delta = K_next - K_raw

        R_E  = abs(H(K_next) - targets.H0) / (1 + abs(targets.H0))
        R_P  = norm(Ptot(K_next) - targets.P0) / (1 + norm(targets.P0))
        R_Q  = norm(Qcm(K_next) - targets.Q0) / (1 + norm(targets.Q0))
        R_L  = norm(AngMom(K_next) - targets.L0) / (1 + norm(targets.L0))
        R_Pi = norm_weighted(Delta) / (1 + norm_weighted(K_next))

        R = (params.aE*R_E + params.aP*R_P + params.aQ*R_Q
             + params.aL*R_L + params.aPi*R_Pi)

        W = R / (1 + R)
        Gamma = Gamma_rule(n, K_next, params)
        S = S / (1 + Gamma * W * h)

        K = K_next
        history.append(K)

    C = closure_defect(history[-1], K0, params.symmetry_group)
    return S, C, history
```

C References and Further Reading

The bibliography below lists the works most directly relevant to this note. Some are used as mathematical anchors; others provide benchmarks or methodological parallels.

References

[1] R. de la Llave, A. González, A. Jorba, and J. Villanueva, "KAM theory without action-angle variables," *Nonlinearity* **18** (2005), 855-895.

[2] A. Chenciner and R. Montgomery, "A remarkable periodic solution of the three-body problem in the case of equal masses," *Annals of Mathematics* **152** (2000), 881-901.

[3] A. A. Trani, N. W. C. Leigh, T. C. N. Boekholt, and S. Portegies Zwart, "Isles of regularity in a sea of chaos amid the gravitational three-body problem," *Astronomy & Astrophysics* **689** (2024), A24.

[4] E. Hairer, C. Lubich, and G. Wanner, *Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations*, 2nd ed., Springer, 2006.

[5] K. R. Meyer, G. R. Hall, and D. Offin, *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, 2nd ed., Springer, 2009.

[6] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry*, 2nd ed., Springer, 1999.

[7] P. M. Austin, "Recursive Survival Geometry: A Phase-Space Information Model for Light-like and Matter-like Propagation," independent manuscript, 2026.

[8] P. M. Austin, "Circular Phase Geometry Without Trigonometric Parametrisation: A Recursive Algebraic Construction," independent note, 2026.